TEXT-BOOK

OF

MECHANICS

BY

LOUIS A. MARTIN, Jr.

Professor of Mechanics, Stevens Institute of Technology

Vol. III.

MECHANICS OF MATERIALS

FIRST EDITION

FIRST THOUSAND

NEW YORK

JOHN WILEY & SONS

London: CHAPMAN & HALL, Limited

1911

Stanhope Press
F. H. GILSON COMPANY
BOSTON, U.S.A

PREFACE

THIS book is a text-book, and not a treatise, on mechanics of materials. Although calculus is freely used it is written for beginners who have had some previous training in theoretical mechanics.

Most, if not all, of the exercises should be solved when they appear in the text, for much matter of importance is contained therein.

An attempt has been made to produce a book which will encourage the student to think, and not to memorize, to do and not simply to accept something already done for him, but which still furnishes sufficient material in the way of explanation and example so that he will not become discouraged.

In its preparation numerous works in English, French, and German have been consulted. No attempt has been made to include the theory of the stresses in curved bars, plates, rotating disks, etc., for these the reader is referred to advanced works; such as,

Grashof, F., Theorie der Elasticität und Festigkeit;
Résal, J., Résistance des Matériaux;
Bach, C., Elasticität und Festigkeit;
Die Maschinen-elemente;
Morley, A., Strength of Materials;
Love, A. E. H., Theory of Elasticity.

I take this opportunity to thank my wife, Alwynne B. Martin, for assistance in the preparation of the manuscript and for reading the proofs; also Prof. R. F. Deimel for suggestions and for reading both manuscript and proof.

L. A. M., Jr.

Castle Point, Hoboken, N. J.
July, 1911

CONTENTS

MECHANICS OF MATERIALS

INTRODUCTION

CHAPTER I

SIMPLE STRESSES

SECTION I

Normal Stress and Strain; Shear

SECTION II

Applications of Hooke's Law

CHAPTER II

Stresses in Beams

SECTION III

Bending Moments and Shearing Forces

SECTION IV

Theory of Simple Bending

SECTION V

Investigation and Design of Beams for Bending

SECTION VI

Shearing Stresses in Beams

CHAPTER III

Deflection of Beams Due to Simple Bending

SECTION VII

The Differential Equation of the Elastic Curve and Its Application

SECTION VIII

The Differential Equations of Beams

CHAPTER IV

Statically Indeterminate Beams

SECTION IX

Propped and Built-in Beams

PAGE

SECTION X

Continuous Beams

CHAPTER V

STRUTS AND COLUMNS

SECTION XI

Eccentric Longitudinal Loads (Short Columns)

SECTION XII

Buckling (Long Columns)

CHAPTER VI

TORSION

SECTION XIII

Stress and Strain Due to Torsion

MECHANICS OF MATERIALS

INTRODUCTION

In every structure various members or parts are joined together and transmit forces.

When the members of a structure are so joined as to permit of relative motion the structure is called a **machine,** and the study of the forces transmitted necessitates the application of the principles of statics, kinematics, and kinetics.

When the members of a structure are so connected that their relative positions must remain unaltered the structure is called a **structure,** in a narrower sense, and the forces transmitted from member to member can generally be determined by means of the principles of statics alone.

Under the general heading of **theoretical mechanics** some of the forces acting upon the various members of certain machines and structures have already been discussed, together with the principles necessary for their determination. During these discussions **the members** of the structures **were**, however, **considered rigid;** they were assumed to remain unaltered in shape irrespective of their dimensions and of the magnitude of the forces to which they were subjected. The materials of which these members must be made cannot fulfill these conditions.

For a member of a certain size and material it is possible to assign certain limiting forces beyond which it is not advisable or even safe to expose this member. Again, given the forces that any member must withstand, it becomes necessary to determine the shape and dimensions of that member which will enable it to withstand said forces with safety, or with appropriate stiffness, or with least weight of material. These, then, are the problems investigated under the subject of **mechanics of materials. The behavior of material under the action of force is our subject matter.**

No advance can be made in this subject by purely mathematical deductions founded upon the principles of mechanics. It is absolutely necessary to introduce the **results of experiments** upon actual materials of construction; the physical properties of the materials used must be considered. In this text we shall assume such physical constants as may be necessary (always using average values, for the physical constants vary between rather wide limits even for the same material) and confine ourselves to the mathematical side of the subject.

CHAPTER I

SIMPLE STRESSES

SECTION I

NORMAL STRESS AND STRAIN; SHEAR

Normal Stress. — Consider, as the simplest case, a bar (weight neglected) of constant section acted on only by the forces F, uniformly distributed over its ends (Fig. 1). This bar is evidently in equilibrium under

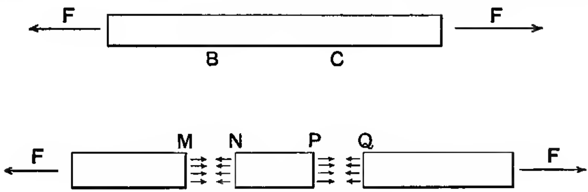


FIG. 1

the **axial forces.** We wish to consider more closely the effect of this load upon the material of the bar. Imagine the bar divided into three parts by planes perpendicular to its axis at B and C. In making free bodies of these three parts forces must be shown at the sections M, N, P, and Q. These molecular forces may be considered as due to cohesion. They are distributed forces, in this case of constant intensity, so that their resultant per square inch of section is constant. From the equilibrium of every portion of the bar it is evident that the resultant of the distributed forces on each of the sections M, N, P, and Q must be equal to F.

This bar is said to be **stressed**; as shown in Fig. 1, the bar is in **tension**; if the forces F were both reversed the bar would be in **compression.** In either case the total stress would be measured by the force F.

Stress. — The unit stress, or simply **the stress,** is the intensity of stress, i.e., the total stress divided by the sectional area of the bar over which the stress is uniformly distributed, so that if A is the sectional area in square inches and F the total stress in pounds, then the unit stress or simply the

$$\textbf{stress} = \mathbf{p} = \frac{\mathbf{F}}{\mathbf{A}} \text{ pounds per square inch.}$$

Strain. — When a bar is stressed it invariably changes in length. This deformation is called the **total longitudinal strain.** Instead of considering the total deformation, it is usual to specify the deformation per unit of length, and this is called the unit strain. Thus if a bar l inches long is put in tension and lengthens Δl inches, the unit longitudinal strain or simply the

$$\textbf{longitudinal strain} = \mathbf{s} = \frac{\Delta \mathbf{l}}{\mathbf{l}}.$$

Relation between Stress and Strain. — The relation between stress and strain can be established only by experiment. The machines used for this purpose are admirably described in Marten's Handbook of Testing Materials.*

* English Translation by G. Henning. Wiley & Sons, N. Y. C.

The diagramatic sketch, Fig. 2, will illustrate the general principle involved. By means of hydraulic pressure at A the test piece BC is put in tension, the force applied being measured by means of the weight W. The length of a portion l of the test piece, well away from its ends,

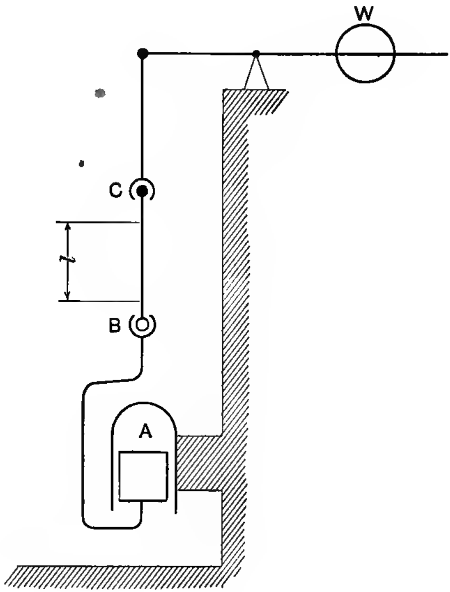


FIG. 2

is measured (by means of micrometer screws) under various conditions of loading. Thus the total stress and the corresponding total longitudinal strain are determined experimentally.

The results of experiments such as described above can best be represented graphically. Consider, as an example, wrought iron. The curve obtained by plotting unit

stress and unit strain as ordinates and abscissas is represented in Fig. 3. Fig. 3 shows that the strain is proportional to the stress up to the point marked *E.L.* This point is known as the **elastic limit.** If the stretching force be diminished to zero at any time before the elastic limit is reached, the test piece will return to its

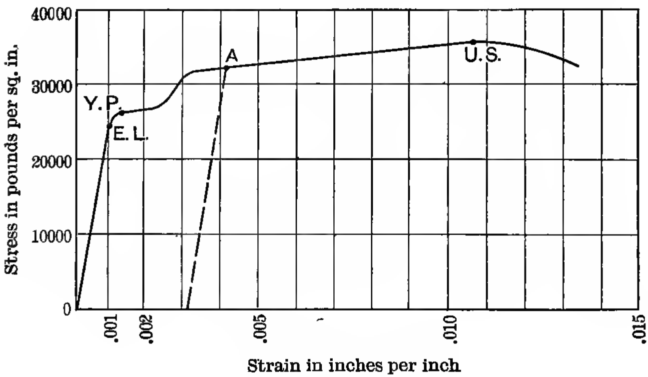


FIG. 3

original length; that is, the strain will wholly disappear, and in plotting the results the line of return will follow the line *E.L.* to *O*.

The point marked *Y.P.* at which a sudden lengthening of the test piece occurs without an increase in the load is called the **yield point.** In a test, as the load upon the test piece is gradually increased, the yield point of the material is conspicuously indicated by a sudden drop of the weight *W*. The yield point is thus readily determined and is sometimes called the commercial elastic limit.

If the loading is carried to say *A* and then interrupted and diminished, the curve of return will be practically parallel to the line o, *E.L.*; and the bar will suffer a permanent elongation, or set, of .003 of an inch per inch of length. On reloading the bar the graph will follow the dotted line to *A* very closely, and a new elastic limit and yield point somewhat higher than *A* will be established.

The ordinate of the point marked *U.S.* denotes the **ultimate strength** of the specimen. At this point the strain increases without additional load or even under slightly diminished load until rupture occurs. The ultimate strength is defined as the maximum load the bar can stand divided by its **original** sectional area.

We are now prepared to state **Hooke's Law,** — that **within the elastic limit the stress is proportional to the strain**; that is, $\frac{\text{stress}}{\text{strain}}$ is constant for any given material. This constant is known as **Young's modulus** or **the modulus of elasticity.** This modulus we shall denote by *E*.

Thus in our notation

$$\mathbf{E} = \frac{\mathbf{p}}{\mathbf{s}},$$

where p = stress in pounds per square inch and s = strain in the direction of the stress p in inches per inch of length; ∴ E is measured in pounds per square inch.

Exercise 1. What are the dimensions of stress, strain, Young's modulus?

Exercise 2. A copper wire .04 inch in diameter and 10 feet long stretches .289 inch under a pull of 50 pounds. Find its modulus of elasticity.

EXERCISE 3. A wrought-iron rod 2 inches square and 10 feet long was lengthened .03 inch by suspending a load from its lower end. Find the load. ($E = 25{,}000{,}000$.)

EXERCISE 4. A wrought-iron tie-rod $\frac{3}{4}$ inch in diameter lengthened $\frac{3}{8}$ inch under a tension of 5000 pounds. How long was it?

EXERCISE 5. How much will a hundred-foot steel tape, $\frac{1}{2}$ inch wide and $\frac{1}{50}$ inch thick, stretch under a pull of 50 pounds? ($E = 30 \times 10^6$ pounds per square inch.)

Changes in Section and Volume. — It has been shown by experiment that longitudinal strain is always accompanied by a lateral strain, so that as the length increases the diameter decreases, and vice versa. Also this lateral strain is proportional to the corresponding longitudinal strain, thus, within the elastic limit

$$\frac{\text{unit lateral strain}}{\text{unit longitudinal strain}} = \frac{1}{m}, \text{ a constant;}$$

$\frac{1}{m}$ is called **Poisson's ratio** and is about .3 for metals.

Thus a rectangular bar, length l, width b, and depth d, will under tension lengthen to say l', while its lateral dimensions diminish to b' and d' respectively (Fig. 4).

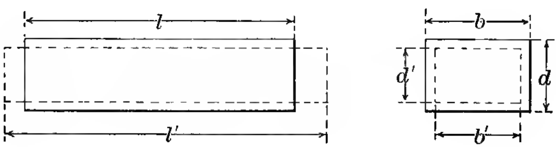


FIG. 4

Then if s represents the longitudinal strain

$$l' = l + ls = l\,(1 + s),$$

PHYSICAL CONSTANTS

Average Values

Material	Elastic Limit	Ultimate Strength			Modulus of Elasticity	Shearing Modulus of Elasticity	Factors of Safety			Poisson's Ratio
		Tension	Compression	Shear			Steady Loads	Variable Loads	Shocks	
	Lbs. per sq. in.	Lbs. per sq. in.	Lbs. per sq. in.	Lbs. per sq. in.	Lbs. per sq. in.					
Structural steel	30,000	65,000	65,000	50,000	30,000,000	11.5×10^6	4	6	10	.297
Wrought iron..	25,000	54,000	40,000	40,000	28×10^6	10×10^6	4	6	10	.277
Cast iron......	3,000	22,500	95,000	20,000 along fiber	12×10^6	5×10^6	8	12	20	......
Timber........	3,000	10,000	7,000	1,200	1.7×10^6	140×10^3	8	10	15	......
Brick (hard)...	....	500	3,000	1,000	....	....	15	25	40	......
Granite........	....	....	20,000	....	....	....	15	25	40	......
Concrete......	....	....	2,500	....	3×10^6	....	15	25	40	......

and if $\frac{1}{m}$ represents Poisson's ratio,

$$d' = d - d\left(\frac{1}{m}\right)s = d\left(1 - \frac{s}{m}\right),$$

$$b' = b - b\left(\frac{1}{m}\right)s = b\left(1 - \frac{s}{m}\right).$$

Also the strained sectional area becomes

$$d'b' = db\left(1 - \frac{s}{m}\right)^2,$$

and as s is very small, the term containing s^2 may be neglected, so that

$$d'b' = db\left(1 - \frac{2s}{m}\right)$$

very nearly, and the decrease in area is $\frac{2sdb}{m}$.

Exercise 6. Show that the elastic change in volume of the above bar under compression is $s\left(1 - \frac{2}{m}\right)ldb$.

Exercise 7. A bar of structural steel, 2.5 inches in diameter and 18 feet 6 inches long, is put under a tension of 64,000 pounds. Compute the change in length, sectional area, and volume. $\left(\frac{1}{m} = \frac{1}{3};\ E = 30{,}000{,}000.\right)$

Factors of Safety. — As already stated, the

$$\textbf{ultimate strength} = \frac{\textbf{maximum or breaking load}}{\textbf{original sectional area}}$$

of the test piece.

The

$$\textbf{working stress} = \frac{\textbf{ultimate strength}}{\textbf{factor of safety}}.$$

This factor of safety is used to bring the working stress well within the elastic limit and to allow for any unforeseen flaws or irregularities in the material used.

It is evident that for strength and durability the stress in any member of a structure must remain below the elastic limit of the material of which it is composed, as otherwise a permanent set or even destruction would result; and as in commercial tests the elastic limit is seldom accurately ascertained, the ultimate strength, with a suitable factor of safety, is used instead. In the case of some materials, such as brick and stone, which are at best of uncertain uniformity throughout, the factor of safety is taken much larger than in the case of steel or wrought iron. The selection of the proper factor of safety is largely a matter of practical experience. (Page 9.)

EXERCISE 8. A short wooden post is 6 inches in diameter. What compressive load can it bear with a factor of safety of 8?

EXERCISE 9. To what height can a hard brick wall of constant thickness be safely carried if it supports its own weight only? Assume the brick to weigh 125 pounds per cubic foot.

EXERCISE 10. The maximum steam pressure in a steam-engine cylinder is 120 pounds per square inch and the piston area is 200 square inches. Find the diameter of the steel piston-rod if lateral bending is prevented.

EXERCISE 11. Short, square wooden columns supporting a platform each carry a load of 16,000 pounds and rest upon foundations of brick. Find the dimensions of the columns and the size of the square column footings if they are needed.

Shear. — The stresses just considered (tension and compression) are called **normal stresses.** Here the tendency is to separate the molecules from each other

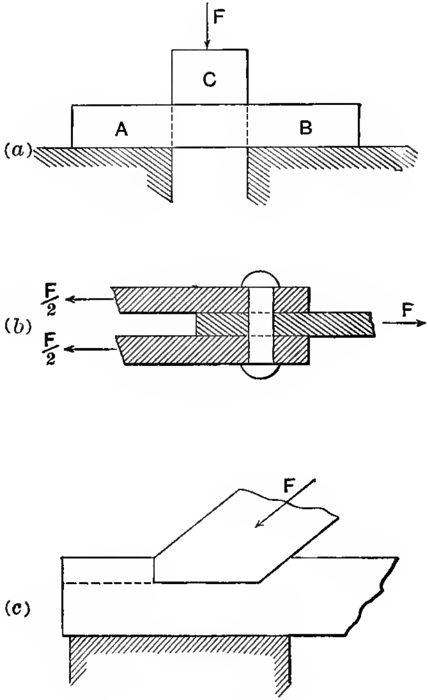


FIG 5

in a direction at **right angles to the plane upon which the stress** is considered.

It often happens that the **forces** acting **tend to force the molecules past each other in a direction parallel to the resisting plane** or section. An action of this kind is called a **shear.** Thus in Fig. 5 (a), if the test piece *A B* rests upon two fixed supports and the block C (which just fits between these supports) is forced downward, the piece AB will be sheared along the planes indicated by the dotted lines. In this case the **area subject to shear is twice the sectional area of the bar** and the total shearing force is F.

As before, the unit shearing stress or simply the

$$\textbf{shearing stress} = \mathbf{q} = \frac{\textbf{total shearing force}}{\textbf{area subject to shear}}.$$

Other illustrations of material subject to shearing stress are shown in Fig. 5; (b) shows a riveted joint and (c) a rafter and tie rod in a roof frame; the dotted lines indicate the planes of shear.

As a numerical illustration, consider the stresses in a wooden test piece, (Fig. 6). The larger cylindrical ends

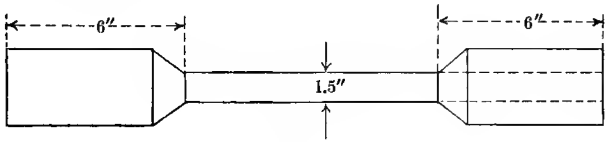


FIG. 6

are clamped in the testing machine and the force required to rupture the specimen somewhere in the thinner section is sought. When this test piece is put in tension two things may happen. Either the piece ruptures under normal stress along a plane perpendicular to its axis, or a cylindrical portion 1.5 inches in diameter is pulled out of the larger end pieces, the specimen thus failing under shearing stress.

Assuming the ultimate strength of wood in tension as 10,000 pounds per square inch, and as the ultimate strength for shear along the grain may be as low as 500 pounds per square inch, we have the tension for rupture under normal stress

$$= \frac{\pi (1.5)^2}{4} \times 10,000 = 17,700 \text{ pounds},$$

while the tension which will cause failure by shearing

$$= \pi (1.5 \times 6)\, 500 = 14,100 \text{ pounds}.$$

Thus the piece is likely to fail by shearing.

EXERCISE 12. To what length should the dimension above represented by 6 inches be increased to assure failure by normal stress?

EXERCISE 13. Calculate the force required to punch a hole 1 inch in diameter through a wrought-iron plate $\frac{3}{8}$ inch thick.

EXERCISE 14. How much will a steel punch 2 inches square and 4 inches long be shortened by the force required to punch a 2-inch-square hole through a wrought-iron plate $\frac{1}{4}$ inch thick?

EXERCISE 15. The diameter of a wrought-iron bolt is $\frac{3}{4}$ inch. What should be the depth of the bolt-head in order that the bolt be equally strong in tension and in shear?

SECTION II

APPLICATIONS OF HOOKE'S LAW

Temperature Stresses. — Materials usually expand with an increase in temperature. The coefficient of expansion is the increase in length per unit of length for an increase of 1 degree Fahrenheit in temperature. Thus given the coefficient of expansion, we can calculate the change in length corresponding to a given change in temperature.

If a rod is prevented from changing its length, the forces resisting this change will stress the rod, and this stress can be calculated by means of Hooke's Law.

EXERCISE 16. A wrought-iron rod 20 feet long and 2 inches in diameter is screwed up to a tension of 9000 pounds. If the temperature falls 10° F., what is the tension in the bar? (Coefficient of Expansion .0000068.)

EXERCISE 17. In electric railways the steel rails are often welded together. Assuming that there are no expansion joints and that no buckling occurs, what is the greatest range of temperature for which the stress will remain within the elastic limit? (Coefficient of Expansion .0000065.)

EXERCISE 18. A battery of boilers is connected by a steel pipe in which no provision was made for expansion. If the temperature of the room is 80° F. and that of the steam 380° F., what stress would result if no change in length occurs?

Resilience. — Resilience is the energy stored in stressed material.

The work done in stretching a bar within the elastic limit is evidently the product of the average force by the total elongation, or $\left(\frac{F}{2}\right)(\Delta l)$. In general the work $W = \int_0^{\Delta l} F dx$ where F is the variable force applied to the rod; F must be expressed in terms of x before integrating for W.

EXERCISE 19. Show that the resilience per unit volume for any substance is $\frac{p^2}{2E}$ if p lies within the elastic limit.

In what units is this quantity measured?

EXERCISE 20. How much work is done in stressing a wrought-iron bar 4 inches in diameter and 54 inches long from 6000 pounds per square inch to 12,000 pounds per square inch?

EXERCISE 21. Find the horse-power required to produce a tension of 56,000 pounds in a steel rod 3 inches in diameter and 68 inches long, if it is stretched 1,200 times per minute.

The lengthening of a cylindrical bar due to its own weight furnishes an interesting application of calculus. (Fig. 7.)

Let w = the weight of the bar per unit volume,
A = its sectional area,
$d(\Delta l)$ = the increase in length of an element,
dx = length of this element,
Δl = total elongation of the bar,
l = length of bar.

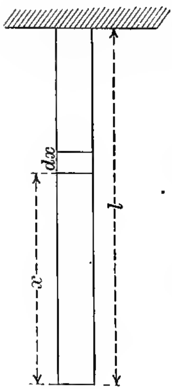


FIG. 7

Then at any point x inches from the lower end the strain

$$s = \frac{d(\Delta l)}{dx},$$

but

$$s = \frac{p}{E} = \frac{Axw}{AE},$$

thus

$$d(\Delta l) = \frac{wxdx}{E},$$

$$\Delta l = \frac{1}{E}\int_{x=0}^{x=l} wxdx = \frac{wl^2}{2E}.$$

EXERCISE 22. Show that the elongation of a cylindrical rod due to its own weight is that produced by a load equal to one-half the weight of the rod applied at the end of a weightless rod, otherwise possessing the physical properties of the original rod.

To find the shape of a rod of constant strength when in tension under its own weight and a load at its end.

By constant strength is meant that every part of the body is to be stressed to the same extent. In the case of a cylindrical rod the material at the top bears the greatest stress (due to the load plus the weight of the rod); this gradually diminishes towards the lower end.

In Fig. 8 let the sectional area of the required rod, at a distance x from the lower end, be A, a variable; P, the load; w, the weight per unit volume; p, a constant, the stress at any point in the rod; and A', the area at any point between $x = 0$ and $x = x$.

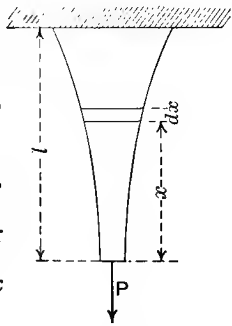


FIG. 8

Then $pA = \int_0^x wA'dx + P$. Also at a section $x + dx$ from the lower end $p(A + dA) = \int_0^x wA'dx + wAdx + P$.

By subtraction

$$pdA = wAdx,$$

or

$$\log_e A = \frac{w}{p}x + C.$$

If A_0 is the area at the lower end, then $A_0 = \frac{P}{p}$, and $A = A_0$ when $x = 0$

$$\therefore \qquad \log_e \frac{A}{A_0} = \frac{w}{p}x,$$

or
$$A = \frac{P}{p} \epsilon^{\frac{w}{p} x}.$$

This equation gives the area of the section of the rod at any distance x from its lower end.

EXERCISE 23. Show by integration that the weight of the above rod is $pA_0 \left\{ \epsilon^{\frac{wl}{p}} - 1 \right\}$.

EXERCISE 24. Find the total elongation of the above rod.

EXERCISE 25. Show that the resilience of a rod, of constant strength, in tension is

$$\frac{Pp^2}{2\,wE}\left(\epsilon^{\frac{wl}{p}} - 1\right).$$

EXERCISE 26. A granite pier is to be 50 feet high and carry a variable load of 100 tons; find the areas of the top and bottom sections if the stress on all sections is to be constant.

EXERCISE 27. In practice the pier in Ex. 26 would not be shaped for constant stress but would be given a trapezoidal vertical section corresponding to the areas calculated in Ex. 26. Under these conditions what would be the stress at the bottom section?

Tension Due to Impact. — The stress due to a load suddenly applied is greater than that due to the same load gradually applied.

Assume the weight W (Fig. 9) to fall from a height h upon a collar at the end of the rod. The stress due to the impact can be calculated by the principle of work as follows.

Let Δl, be the elongation due to the impact,
p, the greatest stress produced,
A, the area of the section of the rod, and
l, its length.

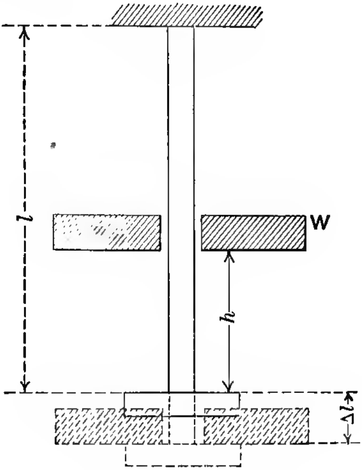


FIG. 9

Then the work done by the falling weight equals the energy stored in the stressed bar, or

$$W(h + \Delta l) = \frac{1}{2}(pA)\left(\frac{pl}{E}\right),$$

and as

$$\Delta l = \frac{pl}{E},$$

$$W\left(h + \frac{pl}{E}\right) = \frac{p^2 lA}{2E}.$$

If the load is suddenly applied (not dropped), then $h = 0$ and

$$p = 2\frac{W}{A}.$$

Hence the greatest stress produced by a sudden application of the load is double the stress due to the load gradually applied.

EXERCISE 28. Find the greatest stress due to the weight W, Fig. 9, when dropped from a height h upon the collar. Why is this greater than $2\frac{W}{A}$?

CHAPTER II

STRESSES IN BEAMS

SECTION III

BENDING MOMENTS AND SHEARING FORCES

Definitions. — If a straight bar under axial load is in tension, or compression, it is called a tie or a strut, respectively. When the lines of action of the forces acting upon a straight (or very nearly straight) bar are perpendicular to the axis of the bar, the bar is called a **beam.** As the loads, and therefore the reactions, acting upon beams are frequently vertical and the axis of the beam is then horizontal, we shall usually assume these conditions, although the methods are equally applicable under any direction of loading. Further, the forces acting upon the beam will all be assumed to lie in the same plane.

Beams are usually classified according to the nature of their supports. Thus a beam supported at one point only, Fig. 10 (a), or the portion of a beam overhanging its supports (b), is known as a **cantilever beam.** When the beam simply rests upon supports at its ends it is called a **simple beam,** Fig. 10 (c). A beam not only supported, but also firmly fixed at both its ends, Fig. 10 (d), is called **a built-in or fixed beam.** A **continuous beam** is one supported at more than two points along its length, Fig. 10 (e).

The loading on a beam is either concentrated or distributed. When a large load is distributed over a relatively short length of a beam its point of application is assumed as bisecting this length, and the load is called a **concentrated load.** A distributed load can be either

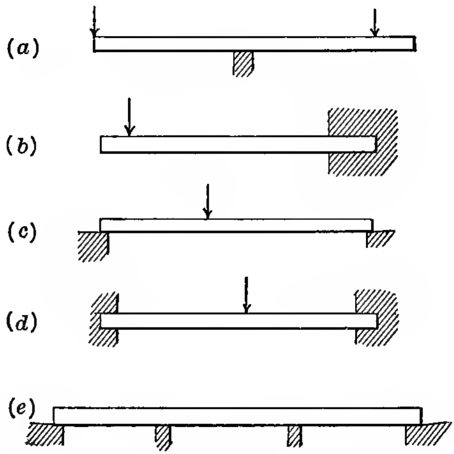


FIG. 10

uniformly distributed so that each element of the length of the beam carries the same load, when it is known as a **uniform load,** or it may vary in any manner.

A beam loaded as above described will be **bent.** The plane of bending will be the vertical plane through the axis of the beam. The elements on the concave side of the beam will be in compression, those on the convex side will be in tension, and thus, evidently, some elements intermediate to these will be neither in tension nor compression. All elements which remain unstressed will form a surface such as AB (Fig. 11), called the

neutral surface; its intersection with the plane of bending, the line AB, is called the **neutral line,** and as it gives the curve into which the beam bends it is sometimes called the **elastic curve.** Any **section** of the beam, such

FIG. 11

as QP, perpendicular to its neutral line, intersects the neutral surface in a line NM; this line is called the **neutral axis of the section.**

Bending Moment and Resisting Moment. — To investigate the stresses which must act in the material

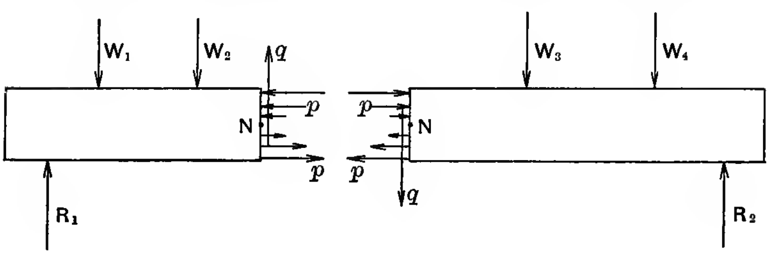


FIG. 12

of a beam under the action of external forces, consider the beam divided into two parts by an imaginary plane perpendicular to its axis, Fig. 12. To preserve the equilibrium of each part under the action of the loads W_1, W_2,

. . . . and the reactions of the supports R_1 and R_2, we have the tensions and compressions (normal stresses) on the elements of the material at the section (indicated by the arrows marked p). Considering the left-hand portion of the beam and noting that for equilibrium the sum of the moments of all forces acting upon it taken about any axis must be zero, we shall find it convenient to use the neutral axis of the section as axis of moments. Thus the sum of the moments of the **external forces** (R_1, W_1, W_2) about N must be numerically equal but opposite in sign to the moments of the **internal forces** (normal stresses, p) also taken about N. This is usually stated by saying that the

bending moment = resisting moment.

EXERCISE 29. Write carefully worded definitions of Bending Moment, External Forces, Internal Forces.

As regards the **signs of the bending moments,** we shall follow the convention of regarding **all clockwise moments acting on the left-hand part of the beam as positive.**

The whole beam being in equilibrium, it follows that the bending moment acting upon the left-hand part of the beam must be **numerically equal** but **opposite in sign** to the bending moment acting on the right-hand part; so that if more convenient the bending moment may be calculated from the right-hand part, but to preserve the convention of signs the **sign of the resulting moment must be changed,** for we shall always consider the bending moment due to the external forces acting upon the left-hand portion of the beam in our discussions.

Shearing Force and Resistance to Shear. — The relation between the bending moment and the resisting moment above deduced is not sufficient for the equilibrium of the left-hand portion of the beam, Fig. 12. We must not forget that the sum of the horizontal and the vertical forces must be separately equal to zero as well.

As it will seldom happen that $R_1 - W_1 - W_2$ is zero, it is evident that for equilibrium some other internal forces besides those due to the normal stresses p must act. The other internal forces on the section must be vertical and therefore due to shearing stresses represented by q in Fig. 12. The introduction of this internal force will not affect our equation of moments, and this resistance to shear must be such as to make the sum of the vertical forces equal to zero. Therefore the total **resistance to shear** at the section considered must be equal numerically but opposite in sign to the sum of all the external forces acting upon the beam to the left of the section considered; this sum is called the **shearing force** at the section considered.

Thus: Shearing Force = Resistance to Shear.

All **upward forces** acting upon the left-hand portion of the beam are to be considered **positive.**

Again, as the beam as a whole is in equilibrium, the sum of the vertical external forces to the left of the section must be equal numerically but opposite in sign to the sum taken on the right. If more convenient the sum of the forces on the right side of the section may be taken for the shearing force, but the sign must then be changed so as to conform with our convention of signs, for the left-hand portion will always be the one considered.

To recapitulate, the bending moment at any section is numerically the sum of the moments of the external forces acting on **either** side of the section about the neutral axis of the section as origin of moments (positive when clockwise), and the shearing force is the sum of the external forces acting on **either** side of the section (positive when upward); **but** as we shall always consider the left-hand portion of the beam **the signs of all calculations made on the right-hand portion must be changed.**

Diagrams of Shearing Forces and Bending Moments. — The variations in the S.F. and B.M. as we pass from one section to another in a beam are best exhibited graphically.

As an example, consider a simple beam carrying a single concentrated load, Fig. 13. The reactions are evidently $2W$ and W.

Assuming the origin at the left abutment, then the S.F. and B.M. at any section at a distance x from the left reaction are for

$$0 < x < l \qquad\qquad l < x < 3l,$$

$$\text{S.F.} = 2W, \qquad \text{S.F.} = 2W - 3W = -W,$$

$$\text{B.M.} = 2Wx, \qquad \text{B.M.} = 2Wx - 3W(x - l)$$

$$= W(3l - x).$$

These equations represent the curves of S.F. and B.M. as plotted in Fig. 13.

Referring to Fig. 13, we see that the concentrated load $3W$ causes a discontinuity in our curves of S.F. and B.M. It will be seen that wherever a concentrated force (load or reaction) acts upon a beam a discontinuity

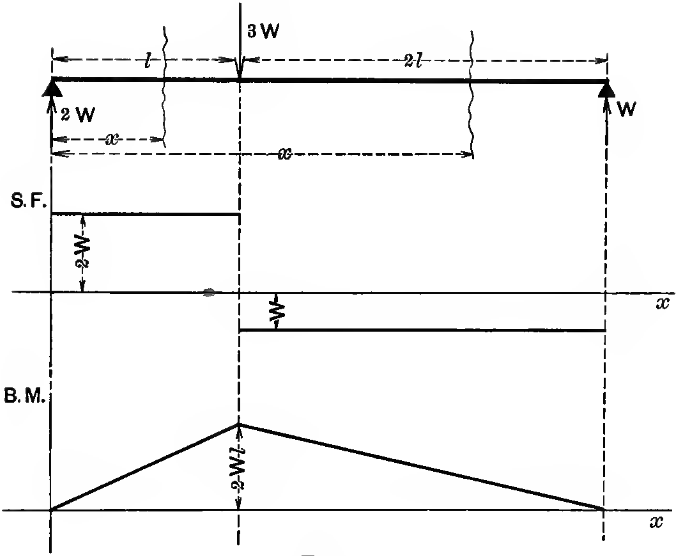


FIG. 13

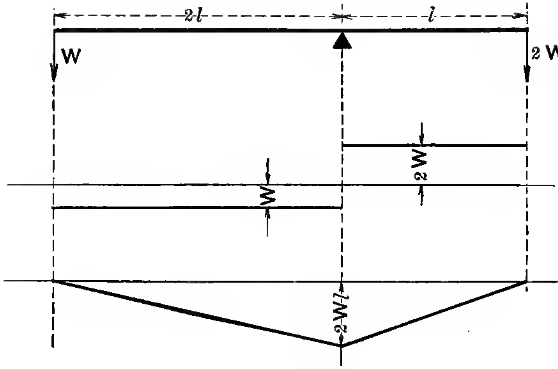
occurs in the S.F. and B.M. diagrams, for at such points the equations formerly expressing the S.F. and B.M. abruptly cease to apply.

EXERCISE 30. Write the equations and plot the diagrams for S.F. and B.M. for the loadings shown in Fig. 14 (a) and (b).

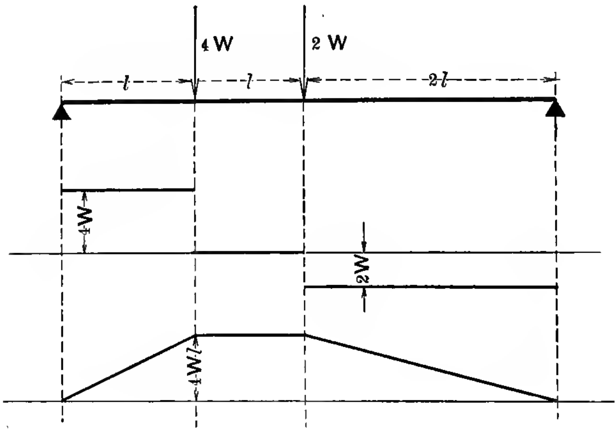
We shall hereafter use Q_x to designate the shearing force at the section x units from the left end of the beam and M_x for the corresponding bending moment.

Consider now the case of a uniformly loaded cantilever beam. (Fig. 15 (a).)

Let w be the load per inch of length, l the length in inches. Here only one interval need be considered, namely, $0 < x < l$; for no abrupt change in loading



(a)



(b)

FIG. 14

occurs. From Fig. 15 (a) we have for any section x inches from the wall

$$\text{S.F.} = Q_x = -[-w(l-x)] = w(l-x),$$

$$\text{B.M.} = M_x = -\left[\{w(l-x)\}\left\{\frac{l-x}{2}\right\}\right] = -\frac{w(l-x)^2}{2}.$$

It should be noticed that in this case it is impossible to find the reaction at the wall, and therefore neither Q_x nor M_x can be found from the forces to the left of the section. The above equations are obtained by considering the forces to the right of the section and carefully following the rules laid down on page 26.

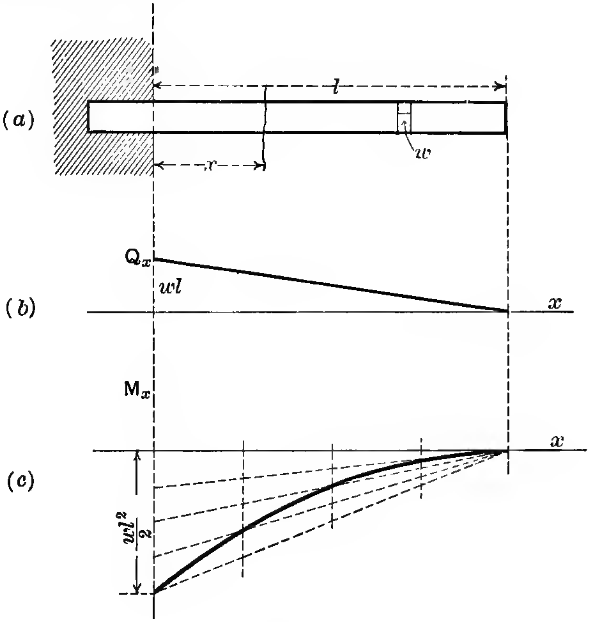


FIG. 15

Plotting these equations as in Fig. 15 (b) and (c), the S.F. diagram is represented by a straight line and the B.M. diagram by a parabola, vertex at $x = l$.

EXERCISE 31. Show that the B.M. diagram for the above case is a parabola with its vertex at $x = l$ and explain and prove its construction as indicated in Fig. 15 (c).

EXERCISE 32. Write equations for S.F. and B.M. and construct geometrically the S.F. and B.M. diagrams for the uniformly loaded beams shown in Figs. 16 and 17.

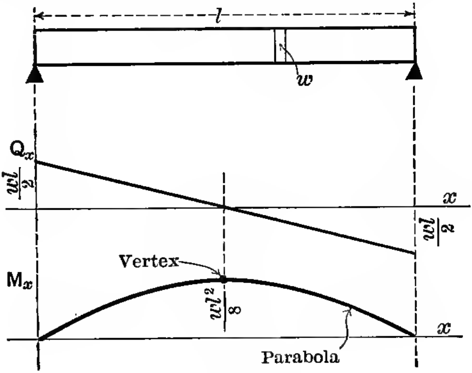


FIG. 16

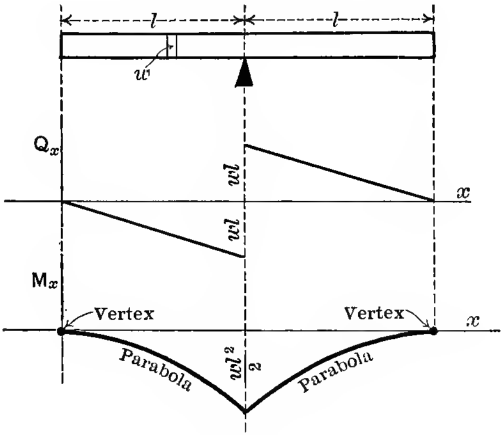


FIG. 17

EXERCISE 33. Write equations for S.F. and B.M. and construct the S.F. and B.M. diagrams for the loadings shown in Figs. 18 and 19.

EXERCISE 34. In the uniformly loaded beam in Fig. 19 find the points at which the B.M. is zero.

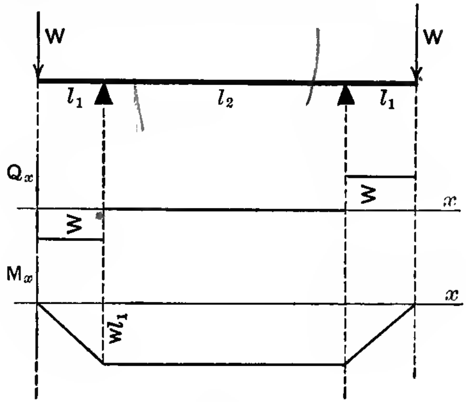


FIG. 18

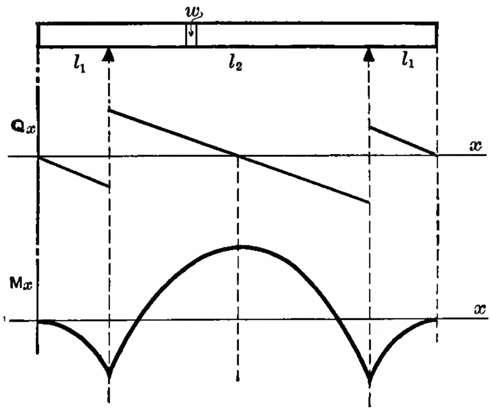


FIG. 19

Relation between Shearing Force and Bending Moment. — The above discussions and diagrams of S.Fs. and B.Ms. show that whenever an abrupt change in the loading occurs a discontinuity in the equations of S.Fs. and B.Ms. results.

In any interval between such discontinuities a definite relation exists between the S.F. and the corresponding B.M.

To establish this relation, we may consider the equilibrium of a portion of any beam between two sections dx apart and x units from the origin, Fig. 20.

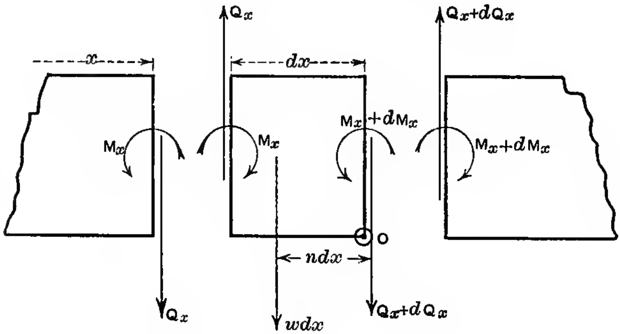


FIG. 20

From the relations

$$\text{shearing force} = \text{resistance to shear},$$

and $$\text{bending moment} = \text{resisting moment},$$

it is evident that we may use Q_x and M_x to indicate the total effect of the internal stresses at the section x units from the origin, and similarly $M_x + dM_x$ and $Q_x + dQ_x$ represent the action of the internal stresses at the section $x + dx$ units from the origin as shown in Fig. 20.

The load on the element is wdx, where w is the rate of loading, and the line of action of the resultant load,

wdx, is located by the distance ndx, where n is some fraction which depends upon the law of change in w. As the element is in equilibrium under the action of the forces shown, we may equate the sum of the moments of these forces about O to zero. Thus, $Q_x dx + M_x - (M_x + dM_x) - wdx\,(ndx) = 0$, and as the differentials of higher order must be omitted we have

$$Q_x dx - dM_x = 0,$$

whence

$$\mathbf{Q}_x = \frac{\mathbf{dM}_x}{\mathbf{dx}}.$$

Thus, in any interval between concentrated loads (or reactions) or between any such loads and any abrupt change in a distributed loading **the shearing force is the first derivative of the bending moment.**

Relation between Rate of Loading and Shearing Force. — Referring again to Fig. 20, we may place the sum of the vertical forces acting upon the element shown as a free body equal to zero.

Thus,
$$Q_x - wdx - (Q_x + dQ_x) = 0,$$

or
$$-wdx - dQ_x = 0,$$

whence
$$-w = \frac{dQ_x}{dx}.$$

The negative sign attached to w, when interpreted by means of Fig. 20, where the loading acts downward, shows that a positive rate of loading would act upward.

Thus if the **sign of the rate of loading,** w, **is taken positive when upward,** then in an appropriate interval **this rate of loading is the first derivative** of the **shearing force,** or

$$\mathbf{w} = \frac{\mathbf{dQ}_x}{\mathbf{dx}}.$$

To recapitulate:

$$w = \frac{dQ_x}{dx} \quad \text{and} \quad Q_x = \frac{dM_x}{dx},$$

so that

$$\mathbf{w} = \frac{\mathbf{dQ_x}}{\mathbf{dx}} = \frac{\mathbf{d^2M_x}}{\mathbf{dx^2}} \, .$$

EXERCISE 35. Sketch a diagram of loading for the beams in Figs. 15, 16, 17, and 19 and note the geometrical relations between the diagrams of loading, shear, and moments. *They are derivative curves.*

A simple beam carrying a total load of P pounds uniformly increasing, as shown in Fig. 21, will serve to illustrate the application of the above principles.

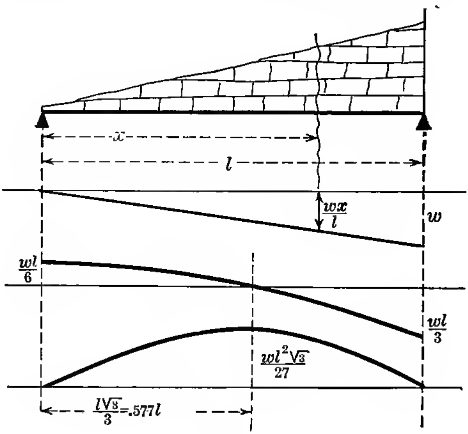


FIG. 21

Here the interval extends from $x = 0$ to $x = l$, as no concentrated loads are applied between the points and **no sudden variation of the distributed load occurs.**

If w pounds per inch represent the rate of loading at $x = l$, then $\frac{wx}{l}$ is the rate of loading at x;

$$\therefore \qquad \frac{dQ_x}{dx} = -\frac{wx}{l},$$

where the minus sign indicates downward loading,

thus, $$Q_x = -\frac{wx^2}{2l} + C_1.$$

The reactions (as can be found by the principles of statics, if we remember that the center of gravity of the whole load is at $x = \frac{2}{3}l$) are $\frac{wl}{6}$ and $\frac{wl}{3}$,

so that $$Q_x = \frac{wl}{6} \quad \text{when } x = 0,$$

and therefore $$C_1 = \frac{wl}{6},$$

thus, $$Q_x = -\frac{wx^2}{2l} + \frac{wl}{6}.$$

And as $$\frac{dM_x}{dx} = Q_x,$$

$$M_x = -\frac{wx^2}{6l} + \frac{wlx}{6} + C_2.$$

C_2 is now obtained by noting that $M_x = 0$ when $x = 0$;

so that, $$C_2 = 0, \text{ and } M_x = -\frac{wx^3}{6l} + \frac{wlx}{6}.$$

It is always important to locate the point at which the maximum bending moment occurs and then to determine its value.

To find the maximum value of M_x, its first derivative should be equated to zero and solved for x. Thus put $Q_x = 0$,

that is $$-\frac{wx^2}{2l} + \frac{wl}{6} = 0,$$

whence $$x = \pm \frac{l}{3}\sqrt{3};$$

of these values the only one applicable to the problem in hand is $x = +\frac{l}{3}\sqrt{3}$. The maximum value of M_x is now obtained by substituting this value of x in M_x; thus,

$$M_{\max} = -\frac{w}{6l}\frac{l^3}{9}\sqrt{3} + \frac{wl}{6}\frac{l}{3}\sqrt{3} = \frac{wl^2}{27}\sqrt{3} = \frac{2Pl}{27}\sqrt{3},$$

for $P = \frac{wl}{2}$.

Notice from Fig. 21 that whenever the B.M. reaches a maximum then the S.F. becomes zero, as must evidently be the case from the relation $\frac{dM_x}{dx} = Q_x$. Also the slope of the B.M. diagram from $x = 0$ to $x = \frac{l}{3}\sqrt{3}$ is positive, and therefore the corresponding part of the S.F. diagram is above the axis, etc.

EXERCISE 36. Show that $M_{\text{greatest}} = \frac{Pl}{3}$ for a cantilever beam l inches long and loaded with P pounds distributed so as to gradually increase from zero at the free end towards the fixed end.

EXERCISE 37. A simple beam l inches long is loaded with a variable, distributed load which gradually increases from zero at one end to w pounds per inch at the center, and then gradually decreases to zero at the other end. Find Q_x, M_x, $M_{\max}$.

Bending Moment by Means of $\int Q_x\,dx$. — The B.M. at any point of a beam can readily be found by means of the relation $\frac{dM_x}{dx} = Q_x$, without recourse to moments. For as $M_x = \int Q_x dx$, if Q_x is expressed in terms of x and the proper constant of integration is determined by means of the existing end conditions, M_x may be found by integration.

To fix our ideas, consider the case of a uniformly loaded simple beam, Fig. 22. $Q_x\,dx$ represents the area of a

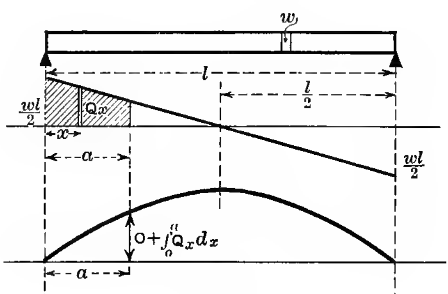


FIG. 22

strip of the S.F. diagram dx in width and the $\int Q_x\,dx$ represents the area of a portion of this diagram bounded by vertical lines. Thus $\int_{x=0}^{x=a} Q_x dx$ represents the shaded area of the diagram, and as S.F. is the derivative of the

B.M., $\int_{x=0}^{x=a} Q_x\, dx$ not only represents the area of the S.F. diagrams as shown in Fig. 22, but also **equals the increase of the ordinate of the B.M. diagram at x = a over its value at x = 0.** As the B.M. at $x = 0$ is zero (the end of the beam being free), $\int_{x=0}^{x=a} Q_x\, dx$ in this case is the ordinate of the B.M. diagram at $x = a$.

EXERCISE 38. Find the maximum B.M. in Fig. 22, without moments or the calculus.

When the integration covers several intervals, the appropriate values of Q_x and proper limits to cover the whole beam, from the left end to the point at which

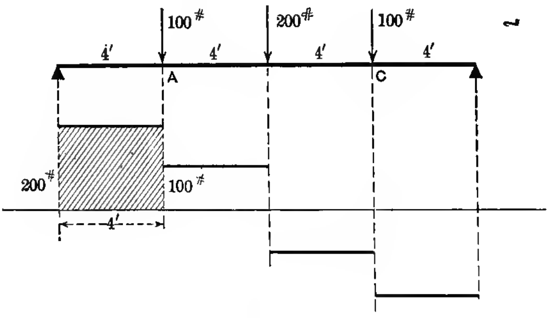


FIG. 23

the moment is sought, must be used. Thus in Fig. 23 the B.M. at A would be found as follows:

$$M_A = \int Q_x\, dx = \int_{x=0}^{x=4} (200)\, dx = 800 \text{ lbs.-ft.}$$

represented by the shaded area of the shear diagram. At C we have

$$M_C = \int_{x=0}^{x=4} (200)\, dx + \int_{x=4}^{x=8} (100)\, dx + \int_{x=8}^{x=12} (-100)\, dx$$
$$= 800 + 400 - 400 = 800 \text{ lbs.-ft.}$$

To what area does this correspond?

EXERCISE 39. A uniformly loaded cantilever 8 feet long, fixed at the left-hand end, carries concentrated loads of 300 pounds at the free end and 200 pounds 5 feet from this end in addition to a distributed load of 100 pounds per foot. Find the B.M. at the fixed end, and at 3 feet from this end.

Draw the S.F. diagram and compute geometrically.

EXERCISE 40. Show that for the loading of Fig. 24 the B.M. at (3) is

$$M_3 = R_1 a + (R_1 - W_1)\, b + (R_1 - W_1 - W_2)\, c.$$

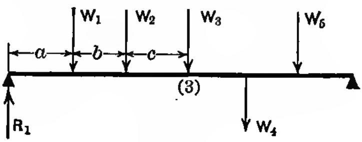


FIG. 24

Dangerous Section. — That section of a beam at which the numerically greatest bending moment occurs is known as the **dangerous section.** This numerically greatest bending moment may occur either at a point of maximum or minimum bending moment or at the point of application of a concentrated load or reaction. Thus in the case illustrated in Fig. 15 the dangerous section is at the wall where the **greatest** negative **bending moment** is $-\frac{wl^2}{2}$; and in the case shown in Fig. 21

the dangerous section is $\frac{l}{3}\sqrt{3}$ inches to the right of the left abutment, where the **maximum bending moment** is $\frac{wl^2}{27}\sqrt{3}$.

As another important case in which the dangerous section occurs at the point of maximum bending moment, consider a **simple beam loaded uniformly over a part of the span only** (Fig. 25).

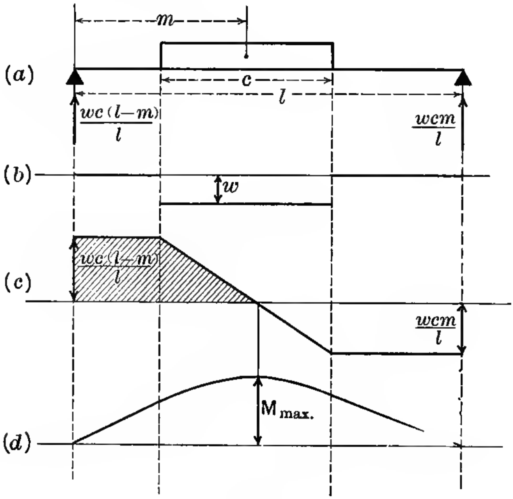


FIG. 25

Let l = span in inches,
m = distance between the left abutment and the center of the load in inches,
c = the length of the load in inches,
and w = load in pounds per inch run.

Then the reactions are, at the left, $\frac{wc(l-m)}{l}$, at the right, $\frac{wcm}{l}$.

In this case three intervals must be considered, for each abrupt change in the load causes a discontinuity in the curves of S.F. and B.M. Such changes occur at $m - \frac{c}{2}$ and $m + \frac{c}{2}$ from the left end. Assuming the origin at the left, the intervals extend from 0 to $m - \frac{c}{2}$, from $m - \frac{c}{2}$ to $m + \frac{c}{2}$, and from $m + \frac{c}{2}$ to l. In Fig. 25, (a) shows the beam, (b) the diagram of loading.

The equations for Q_x and M_x for the three intervals are then as follows:

$$0 < x < m - \frac{c}{2}, \qquad m - \frac{c}{2} < x < m + \frac{c}{2},$$

$$Q_x = \frac{wc(l-m)}{l}, \qquad Q_x = \frac{wc(l-m)}{l} - w\left(x - m + \frac{c}{2}\right),$$

$$M_x = \frac{wc(l-m)}{l}x, \qquad M_x = \frac{wc(l-m)}{l}x - \frac{w\left(x - m + \frac{c}{2}\right)^2}{2},$$

$$m + \frac{c}{2} < x < l,$$

$$Q_x = -\frac{wcm}{l},$$

$$M_x = \frac{wcm}{l}(l - x),$$

and the corresponding diagrams are shown in Fig. 25 (c) and (d).

Evidently the dangerous section occurs in the middle interval and its location can best be found by equating the shearing forces for this interval to zero, whence

$$x = \frac{cl - 2\,cm + 2\,ml}{2\,l}$$

and the maximum bending moment $M_{\max}$ can then be found by substituting this value of x in M_x for the proper interval or by calculating the shaded area of the S.F. diagram (Fig. 25 (c)).

Exercise 41. Find, without the use of the above formulas but by means of geometrical calculations from the shearing force diagram only, the dangerous section and maximum bending moment for a simple beam loaded uniformly over the left-hand half of its length.

Exercise 42. A simple beam, span 30 feet, is loaded with 100 pounds per foot for a distance of 10 feet starting at 15 feet from the left abutment. Find the maximum bending moment and the dangerous section by means of the S.F. diagram.

As already stated, the dangerous section does not always occur at a mathematical maximum of the bending moment. Sometimes a greater value of the bending moment is found at the point of application of a concentrated force.

Exercise 43. A beam 24 feet long overhangs its right abutment by 8 feet, the left abutment being at the one end of the beam. If it is uniformly loaded with 2000 pounds per foot, and carries 16,000 pounds at its right end, calculate the maximum bending moment, the numerically greatest B.M., and find its dangerous section. Draw the S.F. and B.M. diagrams.

Principle of Superposition. — When several loads (concentrated or distributed) act upon a beam the total S.F. or B.M. at any section can be found by adding the S.Fs. and the B.Ms. at that section due to each load considered separately. This is known as the principle of superposition. It will be found especially convenient when concentrated and distributed loads occur simultaneously.

Thus in Fig. 26 the dotted line represents the effect due to the uniformly distributed load, the dotted dashed line the effect of the concentrated load, and the full line their combined effect or the total S.F. and B.M.

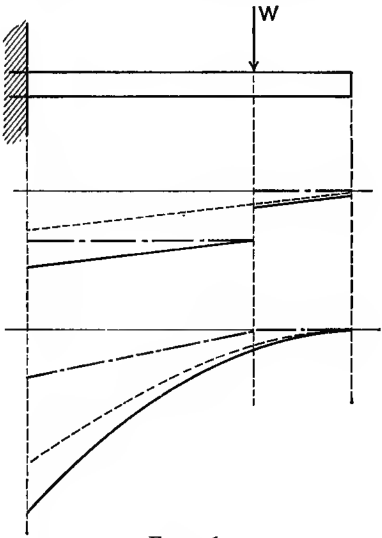


FIG. 26

EXERCISE 44. Sketch diagrams of S.F. and B.M., showing the effect of the loadings shown in Figs. 18 and 19 acting simultaneously.

Bending Moments by Means of the Funicular Polygon. — The bending moments due to concentrated loads can be found graphically by means of the funicular polygon constructed for these loads. In Fig. 27 are shown the loading, the vector polygon, and the corresponding funicular polygon. The **vertical** distance *m* between the closing line of the funicular polygon, *oc*, and *oa* is proportional to the bending moment at *x*.

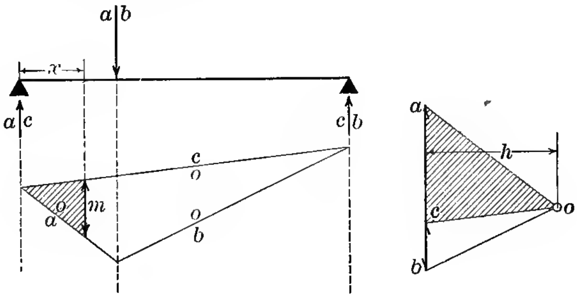


FIG. 27

To show this let the horizontal distance from *O* to the vector polygon be *h*, then as the shaded triangles are similar we have,

$$\frac{m}{x} = \frac{(ca)}{h} \quad \text{or} \quad (ca)x = hm;$$

but (ca) represents the reaction at the left, so that $(ca)x$ represents the B.M. at x. Assume the scale of forces as p pounds to the inch, and the scale of distances q inches to the inch then $(ca)p$ is the reaction in pounds and xq its arm, so that the B.M. at x is $(ca)p \times xq$ and equals $hmpq$, for $(ca)x = hm$. If now m is measured in inches and each inch represents hpq inch-pounds where h is measured in inches, the result would be the B.M. at x.

The scales are p pounds per inch for forces, q inches per inch for distances, and hpq inch-pounds per inch for B.M.

EXERCISE 45. Show that the above construction holds for the third interval of a simple beam carrying four concentrated loads.

EXERCISE 46. If the scale of forces is 100,000 pounds per inch and the scale of distance 12 inches per inch, what would be an appropriate value for h, the polar distance, and what is the corresponding scale of B.Ms.? (A simple scale of moments is to be desired.)

For distributed loads the above construction can be used, provided the load be divided into short lengths and the corresponding weights be considered concentrated at the centers of gravity of these elementary loads. The resulting funicular polygon will then be straight-sided, while the true curve of the bending moment is the **inscribed** curve to this polygon.

SECTION IV

THEORY OF SIMPLE BENDING

Simple bending occurs when a beam is bent by **couples** applied to its ends so that no shearing action takes place. In Fig. 18 the middle interval of the beam is a case in point; here there exist no shearing forces, and the bending moment is constant, being due to the couple formed by the load W and the left reaction, which is numerically equal to W. In Fig. 28 the beam originally straight and of the same cross section throughout (which is also assumed symmetrical with respect to the central longitudinal plane of bending) is acted on by couples whose moments are represented by the curved arrows M. Under these conditions **Bernoulli's assumption, that transverse sections originally plane will remain plane and normal to longitudinal fibers after bending,** seems reasonable. In beams it is convenient to consider the material as consisting of longitudinal fibers or elements

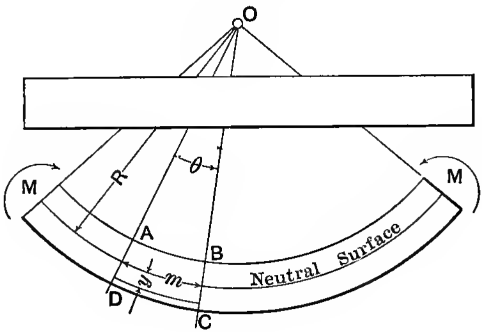


FIG. 28

of infinitesimal section, even though the substance may in no wise be of a fibrous nature. Moreover, it is assumed that these imaginary fibers act independently of each other, so that layers parallel to the neutral surface may expand and contract without hindrance, and **Hooke's Law may be assumed to hold.**

Consider an element of the beam originally inclosed between two transverse planes m inches apart. After bending this is assumed to become the element $ABCD$ whose length on the **neutral surface** is still m, and AD and BC still remain planes intersecting at the **center of curvature,** O, the beam having a **radius of curvature** R to the neutral surface. Let the element subtend an angle θ; then any fiber at a distance y from the neutral surface will have a length $(R + y)\theta$. The original length of this fiber was $m = R\theta$, its elongation is thus $y\theta$ and its strain $s = \frac{y\theta}{R\theta} = \frac{y}{R}$. If the stress due to bending is p pounds per square inch and Hooke's Law is assumed to apply,

then $$E = \frac{p}{s} = \frac{pR}{y} \quad \text{or} \quad p = E\frac{y}{R}$$

where E is Young's Modulus. If E is the same for tension and compression, for the material considered,

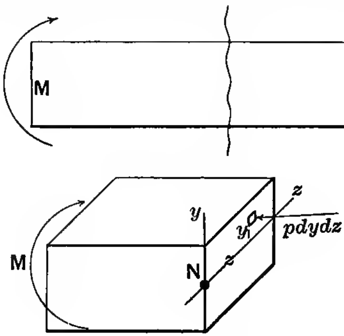


FIG. 29

then the compressive stress at y inches to the other side of the neutral axis will be equal to the stress p above used. Also p increases as y increases and reaches its greatest value at the outermost fibers of the beam. The **stress, p, is said to follow the straight-line law,** being proportional to its distance from the neutral surface.

To study the relation existing between the stresses and the B.M., a portion of the beam must be considered as a free body. In Fig. 29 this is done and N indicates the position of the **neutral axis of the section** considered. The section of a fiber is shown and the internal force (due to the stress p) acting upon it is $p\, dy\, dz$.

As this portion of the beam is in equilibrium, two equations will furnish the necessary conditions.

$$\sum \text{Moments about the neutral axis}$$

$$= M - \int\int y(p\, dy\, dz) = 0, \quad \ldots \ldots \quad (1)$$

the function to be integrated over the section of the beam,

and $$\sum \text{Horizontal Forces} = \int p\, dy\, dz = 0. \quad . \quad (2)$$

For the moment M is due to vertical forces only, and as these form a couple they in turn will balance so far as the sum of the vertical forces is concerned.

From equation (1) it follows that

$$M = \int\int y\,(p\, dy\, dz), \quad \text{and as} \quad p = E\frac{y}{R},$$

$$M = \int\int y\left(E\frac{y}{R}\right) dy\, dz = \frac{E}{R}\int\int y^2\, dy\, dz.$$

The quantity $\int\int y^2\, dy\, dz$ is a **second moment of area,** but by reason of its resemblance to the similar expression already studied in mechanics involving the density of the material, etc., it is often, though incorrectly, called a **moment of inertia,** and the letter I is used to designate it.

Thus $$M = \frac{EI}{R},$$

where M is the B.M. at the section considered,

E is Young's Modulus,

I is the second moment of area about the **neutral axis of the section considered,**

and R is the radius of curvature to which the neutral surface of the beam is bent by the moment M.

As $$E = \frac{pR}{y}, \qquad \frac{E}{R} = \frac{p}{y}$$

and $$\mathbf{M} = \frac{\mathbf{E}}{\mathbf{R}}\mathbf{I} = \frac{\mathbf{p}}{\mathbf{y}}\mathbf{I},$$

where p is the stress due to M at a distance y from the neutral axis and $\frac{\mathbf{p}}{\mathbf{y}}\mathbf{I}$ **is the resisting moment** of the section considered.

As yet the position of the neutral axis so often mentioned has not been located.

Equation (2) above will furnish the requisite condition.

As $$\int\int p\,dy\,dz = 0 \quad \text{and} \quad p = E\frac{y}{R},$$

we have

$$\frac{E}{R}\int\int y\,dy\,dz = 0, \quad \text{or} \quad \int\int y\,dy\,dz = 0.$$

This directs attention to the well-known formula for the center of area of laminas,

$$\bar{y} = \frac{\int\int y\,dy\,dz}{\int\int dy\,dz},$$

where $\bar{y}$ is the distance from the axis of z to the center of area.

Now by the above relations $\bar{y}$ must be zero, and thus the **center of area of the section lies upon the neutral axis,** which is our axis of z, Fig. 29.

Digression as to Centers of Area and the Location of the Neutral Axis. — The above discussion shows that the neutral axis of a beam section must always pass through the center of area of this section, for only under this condition can the resultant force due to the compressive stresses (above the neutral axis) and the resultant force due to the tensile stresses (below the neutral axis) be in horizontal equilibrium of translation.

Thus if we locate the center of area of a beam section the position of the neutral axis is determined.

The well-known equation for locating the center of area of a plane surface is

$$\bar{y} = \frac{\sum y\,(\Delta A)}{\sum (\Delta A)},$$

where $\bar{y}$ is the distance of the center of area from any assumed axis of reference and y is the distance from the center of area of any elementary area, ΔA, from the same axis of reference, and the summation must include all elementary areas composing the area whose center is sought.

In certain (rare) cases the following formula,

$$\bar{y} = \frac{\int\int y\,dy\,dz}{\int\int dy\,dz},$$

corresponding to the above but in the notation of the calculus, may be useful.

It will be found that most engineers call the **center of area** of a beam section its **center of gravity.** This is a misleading use of the term "center of gravity." If the beam section is regarded as a thin lamina of uniform density, then the center of gravity of this lamina will of course coincide with the center of area as above defined.

As an **example** illustrating the locating of the neutral axis of a beam section, consider the section shown in Fig. 30 (a). This is the section of a built-up beam composed of two angles and a plate.

The dimensions, the area, and the location of the center of area of the angles used can be found in the handbooks of the steel companies; these are given in Fig. 30 (b).

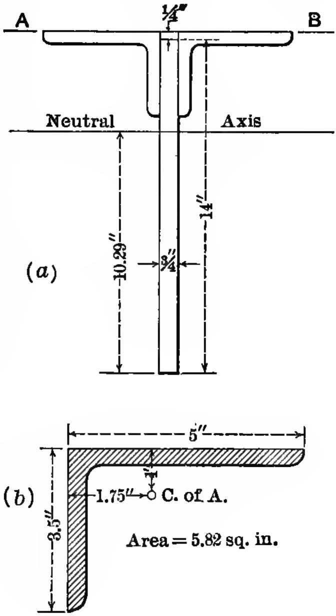


FIG. 30

After assuming any horizontal axis as axis of reference, say AB, from which to measure the y's, the table on the following page may be filled out.

If the loading is normal to AB, the neutral axis of this section is a line parallel to AB and passes through the center of area of the whole section.

In this example the centers of area of both angles are one inch from AB, so that their areas may be added and the two angles considered as one " ΔA."

	ΔA	y	$(\Delta A)\,y$
2 ⌊s 1 plate	11.64 10.50	1 7.25	11.64 76.13
	22.14		87.77

$$\therefore \qquad \bar{y} = \frac{\sum y\,(\Delta A)}{\sum (\Delta A)} = \frac{87.77}{22.14} = 3.96 \text{ inches,}$$

so that the center of area of the section is 3.96 inches below the line AB or 10.29 inches above the lower edge of the plate, and must lie upon the vertical axis of symmetry.

Exercise 47. Locate the center of area of an angle whose long leg is 5 inches, short leg 3 inches, and the width of which is .75 inches.

Exercise 48. Locate the center of area of a deep "channel" formed by two angles and a plate. Use data of Fig. 30 (a) and (b).

Note. — The $\frac{1}{4}$-inch overhang should always be allowed, as in Fig. 30 (a), for practical reasons of construction and erection.

Second Moments of Areas. — The definition of I (the second moment of area, or so-called " moment of inertia ") is $I = \int\int y^2\,dy\,dz$, where the integration must extend over the area considered and y is the distance from the elementary area ($dy\,dz$) to the axis about which I is required

EXERCISE 49. Show that the values of I about the axes indicated in Fig. 31 are correct.

The results indicated in Fig. 31 should be remembered, for they will be of frequent application.

As the values of I here used differ from the moments of inertia of mechanics by a constant factor only, all theorems proved for moments of inertia will apply to the second moments of areas now considered.

Two theorems to be remembered are:

1. The I of a plane area about any axis equals the sum of the I's of all its parts about the same axis.

2. The "moment of inertia" of any plane figure, I, about any axis in its plane equals the "moment of inertia" about a parallel axis through the center of gravity (area) of the figure, I_g, plus the product of the area of the figure, A, and the square of the distance between the parallel axes, l^2.

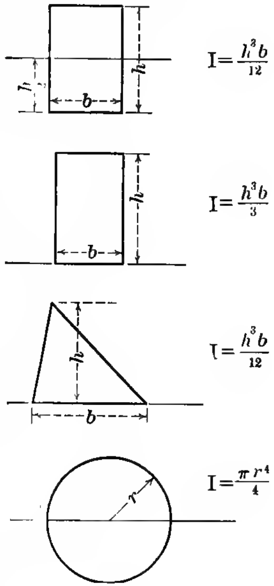


FIG. 31

Thus,

$$\mathbf{I = I_g + Al^2}$$

EXERCISE 50. Prove theorem 2 from the definition of I.

As an application of these theorems, consider the "moment of inertia" of an I section, as illustrated in Fig. 32, about the horizontal neutral axis.

The figure must be divided into parts which conform to the conditions of the known "moments of inertia" as given in Fig. 31. Such parts would be

$$ABCD\left(I = \frac{d^3b}{12}\right),$$

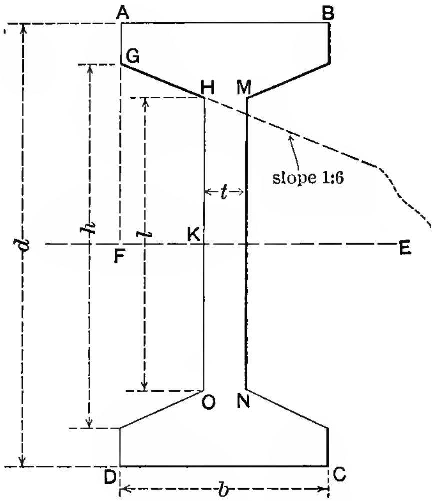


Fig. 32

minus four times $GHKF$. The I of $GHKF$ can be obtained from ΔFGE,

$$\left(I = \frac{\left(\frac{h}{2}\right)^3 (3 h)}{12} = \frac{3\,h^4}{96}\right) - \Delta HKE, \left(I = \frac{\left(\frac{l}{2}\right)^3 (3\,l)}{12} = \frac{3\,l^4}{96}\right),$$

so that the moment of inertia of the whole section about the neutral axis parallel to the flange equals

$$\frac{d^3b}{12} - 4\left\{\frac{3h^4}{96} - \frac{3l^4}{96}\right\} = \frac{1}{12}\left\{bd^3 - \frac{3}{2}(h^4 - l^4)\right\}.$$

EXERCISE 51. Show that I for the section shown in Fig. 32 about a neutral axis parallel to the web (vertical axis) is $\frac{1}{12}[b^3(d-h)+lt^3+\frac{1}{24}(b^4-t^4)]$.

As **another illustration,** consider the section shown in Fig. 30 (a). From the handbooks the I's of the angle shown in Fig. 30 (b) are

I about the neutral axis parallel to long leg = 5.55 ins.^{4}

I about the neutral axis parallel to short leg = 13.92 ins.^{4}

The I of the built-up section is to be found about its horizontal neutral axis 3.96 inches below AB.

	Area of part, A	Distance of neutral axis of A from neutral axis of whole section, l	Al^2	I of A about neutral axis of A, I_g	I of A about neutral axis of whole section
2 ⌊s	11.64	2.96	102.0	11.10	113.1
1 plate	10.50	3.29	113.7	$\frac{h^3b}{12} = 171.4$	285.1

whence the required I of the whole section about its neutral axis } = 398.2

EXERCISE 52. Find two "moments of inertia" for the section described in Ex. 48, one about a neutral axis perpendicular to and the other about a neutral axis parallel to the plate.

EXERCISE 53. Calculate the "moment of inertia" of the section shown in Fig. 33 about the neutral axis parallel to the plate. Assuming the following data concerning the channels (taken from the handbook): Area of section of each channel 6.03 inches^2, I about a neutral axis perpendicular to the 12-inch side 128.1 inches^4, center of gravity .70 inch inside of web measured from the 12-inch side.

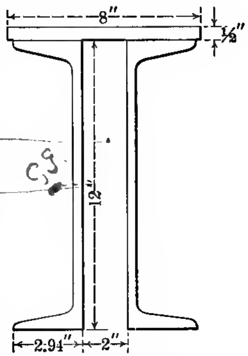


FIG. 33

Section V

INVESTIGATION AND DESIGN OF BEAMS FOR BENDING

The formula for the investigation and design of beams for bending is $M = \frac{p}{y} I$. Here p signifies the stress (pounds per square inch) on a fiber at a distance of y inches from the neutral axis. For the present purpose it is necessary to find or use the stress in that fiber exposed to the greatest stress. Let p_t and y_t be the stress on and the distance from the neutral axis to the fiber at the greatest distance on the tension (convex) side of the beam; then as the stress follows the straight-line law p_t will be the greatest tensile stress in the beam, and the formula

$$\mathbf{M} = \frac{\mathbf{p}_t}{\mathbf{y}_t} \mathbf{I},$$

connects it with the bending moment at the section considered.

Similarly $$\mathbf{M} = \frac{\mathbf{p}_c}{\mathbf{y}_c} \mathbf{I},$$

where the subscript c refers to the compression (concave) side of the beam.

Of course if y_t and y_c are unequal, then the fiber stress at the fiber most remote from the neutral axis must be considered, and if the working stresses of the material for tension and compression are unequal, the least must be selected, other things being equal.

As an **illustration,** find the steady, uniformly distributed load that a wooden cantilever beam 5 feet long, 2 inches broad, and 3 inches deep can carry.

From the table of physical constants, the ultimate strength of wood is 10,000 and 7,000 for tension and compression respectively, and the factor of safety is 8. As the section is rectangular $y_t = y_c = 1.5$ inches, so that the stress on the compressive side must be considered, this being the fiber most likely to fail. Also the section of the beam carrying the greatest B.M. will contain that portion of the extreme compression fiber subject to the greatest stress. In a uniformly loaded cantilever beam this section is evidently at the wall and the B.M. at this section is

$$M = \frac{wl^2}{2} = \frac{w\,(5 \times 12)^2}{2} \quad \text{also,} \quad I = \frac{bh^3}{12} = \frac{2\,(3)^3}{12},$$

$$y_c = \frac{3}{2} \qquad \text{and} \quad p_c = \frac{7000}{8},$$

so that

$$M = \frac{p}{y} I$$

gives

$$\frac{w\,(5 \times 12)^2}{2} = \frac{7000 \times 2 \times 2\,(3)^3}{8 \times 3 \times 12},$$

whence $w = 1.46$ pounds per inch; this includes the weight of the beam.

In this problem the shearing stress at the section considered has been neglected and the formula for simple bending has been assumed to hold. Later the effect of shear will be considered.

EXERCISE 54. A rectangular wooden cantilever beam 10 feet long, 6 inches deep, is to support a load of 300 pounds at its free end. What should be its width, neglecting its weight and assuming a steady load?

Assuming the width above computed and the weight of the wood at 30 pounds per cubic foot, recompute the width including the weight of the beam in the load to be carried by it.

EXERCISE 55. What total uniformly distributed load will a wooden floor beam 2 by 8 inches and 16 feet long carry?

EXERCISE 56. A steel engine shaft rests on bearings 5 feet apart and carries a 4-ton flywheel midway between the bearings. Find the diameter of a shaft of constant section, factor of safety 10.

EXERCISE 57. A hollow, circular, cast-iron beam, outside diameter 6 inches, inside 5 inches, rests on supports 10 feet apart. What load midway between the supports can it carry with a factor of safety of 10? (Neglect the weight of the beam.)

EXERCISE 58. Same as Ex. 57, considering the beam of solid section and 6 inches diameter.

EXERCISE 59. Compare the weights of the beams and the loads supported in Exs. 57 and 58.

When a rectangular beam is to be designed, and neither width, b, nor depth, h, are known, a relation between h and b can be found by means of $M = \frac{p}{y} I$, and then assuming either b or h the other dimension can be determined. For practical considerations h should not exceed (about) $6\,b$.

EXERCISE 60. Wooden floor joists of 14-foot span and spaced 12 inches from center to center are expected to carry a floor load of 80 pounds per square foot. What is a suitable size if the stress is not to exceed 900 pounds per square inch?

EXERCISE 61. One of the joists in Ex. 60 comes at the side of an opening 4 feet by 6 feet. The load on the shorter joists (10 feet long) is partly carried by a joist 6 feet long, one end of which rests on the joist to be considered at 4 feet from its end. How thick should this joist be?

EXERCISE 62. A balcony is to project 6 feet from a wall and be supported by wooden beams spaced 3 feet apart. If the load is to be 200 pounds per square foot and the greatest fiber stress allowed is 800 pounds per square inch, find the size of the beams.

When the material of the beam is steel, then the working stress for both tension and compression is the same, and only the stress on the fiber most remote from the neutral axis need be considered. Under this condition y in $M = \frac{pI}{y}$ represents the distance from the neutral axis to the fiber at the greatest distance from the neutral axis of the section; if this is called c, then the quantity

$\frac{I}{c}$ is called the **section modulus,**

it depending only on the size and shape of the section, and not on the physical properties of the material.

The section modulii of standard rolled sections are listed in the handbooks.

EXERCISE 63. What should be the section modulus of a simple steel I-beam designed to support a uniform load of 500 pounds per foot run, and two equal loads of 2 tons, 4 feet from each end of the span of 20 feet? The stress in steel beams should not exceed 16,000 pounds per square inch.

EXERCISE 64. What is the section modulus of the section shown in Fig. 33? What concentrated load at mid-span could this built-up beam carry on a span of 30 feet?

EXERCISE 65. A Cambria I-beam No. B 21 is 7 inches deep, weighs 20 pounds per foot, has a moment of inertia of 42.2 inches^4, and a working stress of 16,000 pounds per square inch. What total uniformly distributed load can it carry, span 20 feet?

Many different sections evidently have the same section modulus, and it remains for the designer to select that section best suited to the case at hand. It should be noted that the greater the distance of a fiber from the neutral axis the greater is the stress in it due to bending. Thus to use the material to the best advantage the area of the section near the neutral axis should be small and should enlarge where the stress is greatest. The common steel I-beam is an illustration. Also plate girders built up of standard sections take in general the form of an I. Nevertheless (as will be seen later), enough area at the neutral axis must be preserved to sustain the shearing stresses there acting.

Exercise 66. Find the depth and breadth of the rectangular beam of greatest strength that can be cut from a circular log d inches in diameter.

Exercise 67. Show that an error of about 1 % is made in the greatest fiber stress when the weight of the beam is 2 % of a concentrated load at the center of the span and said weight is neglected.

Exercise 68. Three sections of a water-pipe, each 12 feet long, are leaded end to end. In lowering them into a trench where should the two slings be placed so that joints will not be strained? Neglect the extra weight of the sockets.

Exercise 69. Prove that the B.M. in pound-inches at any section of a uniformly loaded simple beam is one-half the load per inch multiplied by the product of the lengths of the two segments into which the section divides the beam, both in inches.

Beams of Uniform Strength. — It is evident that beams designed in the manner just described are too strong for all but one section, — the dangerous section. Therefore, not only is the material at the dangerous section not stressed to its working stress except at the extreme fiber, but also the stresses in the extreme fibers of other sections fall below the working stress.

A beam so designed that the stresses in the **extreme fibers of all sections** are the same and equal to the working strength is called a beam of **uniform strength.**

As the extreme fiber stress depends upon the bending moment, M, at the section as well as upon the corresponding section modulus, $\frac{I}{c}$, the design must depend upon the manner of support and the loading.

Consider the case of a **cantilever supporting a concentrated load at its end** (Fig. 34 (a)). Here $M_x = Wx$, and as $\frac{I}{c} = \frac{bh^2}{6}$, if a rectangular section is assumed, $M = \frac{pI}{y}$ gives for p_c, the extreme fiber stress at the section x inches from the end,

$$p_c = \frac{6\,Wx}{bh^2},$$

as p_c is to be constant and as x varies from section to section either b or h must vary.

Assuming the beam of constant width b and putting the variable depth $h = y$, we obtain

$$p_c by^2 = 6\,Wx,$$

or

$$y^2 = \frac{6\,Wx}{p_c b},$$

the equation of a parabola, vertex at free end of beam. The depth of the beam at the wall, h, can be obtained by placing x equal to l, so that

$$h = \sqrt{\frac{6\,Wl}{p_c b}}\,.$$

Fig. 34 (b) shows an elevation, and (c) a plan of this beam. Of course the upper surface of the beam need not be plane, provided the y's all equal the value indicated above. It should also be noted that the depth of the beam cannot be made zero at $x = 0$. Sufficient area must be left at this section to resist the shear, which is neglected in the above calculation.

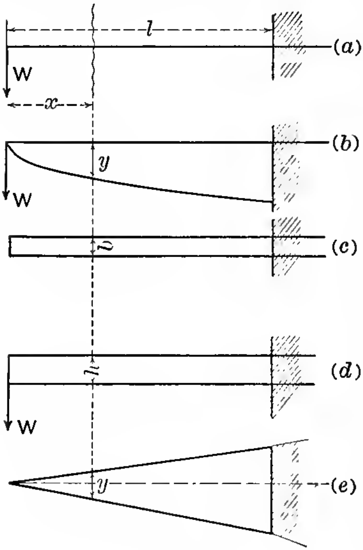


FIG. 34

If the depth of the beam, h, is assumed constant, then the width, y, will vary, and $p_c y h^2 = 6\,Wx$ or $y = \dfrac{6\,Wx}{p_c h^2}$, and the width at the wall is $b = \dfrac{6\,Wl}{p_c h^2}$. See Fig. 34 (d) and (e) for the elevation and plan.

EXERCISE 70. Design a cantilever of uniform strength if it carries a uniformly distributed load.

Test your answers by the theory of dimensions.

Exercise 71. Design a uniformly loaded simple beam of uniform strength and constant width, and calculate the depth at mid-span.

The above design can be used advantageously only when the beams are of cast iron, as in machine frames, brackets, etc.; and due attention must be paid to leave sufficient area at the ends to allow for the shearing stresses (to be considered later). Plate girders made by riveting together standard shapes and plates form approximately beams of uniform strength, but here the sections are no longer rectangular.

Modulus of Rupture. — When the stresses due to bending exceed the elastic limit the formula $M = \frac{pI}{y}$ can no longer be applied, as is evident from the assumptions made in its deduction. Nevertheless, in making transverse tests of wood and cast iron the formula is employed on test pieces of rectangular section and p calculated for the extreme fiber, using the bending moment at rupture for M. The value of p so calculated is called the **modulus of rupture.** It is not a physical constant of the material, and can be used for comparison only on bars of the same length and section.

Exercise 72. A simple beam 6 feet long, 2 inches broad, 3 inches deep, is broken by a weight of 1200 pounds at the center. Find its modulus of rupture.

Section VI

SHEARING STRESSES IN BEAMS

Horizontal Shear. — So far the stresses due to pure bending only have been considered. These stresses were found to be normal stresses (tension and compression) varying according to the straight-line law, and proportional to the distance of the fiber from the neutral axis. Now the effect due to the shearing forces is to be investigated.

As $Q_x = \dfrac{dM_x}{dx}$ (see page 33), a change in the bending moment neccssarily involves a shearing force.

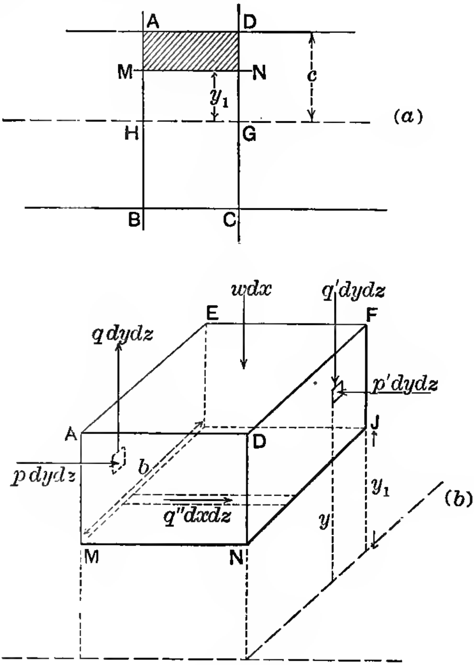


FIG. 35

In Fig. 35 (a) let AB be any section of a beam, DC a section at a distance dx from the first, HG the neutral surface, and MN any plane parallel to the neutral surface. The beam is assumed to be of rectangular section and b inches wide. Fig. 35 (b) shows the element $AMND$ of Fig. 35 (a) as a free body. The forces acting upon it are evidently due to (1) the normal stress on the face ME; (2) the vertical shearing stress on ME; (3) and (4) similar forces on the face NF; (5) the load wdx. No stresses can act upon the faces AN, AF, and EJ, as these are parts of free surfaces of the beam. On the face MJ no vertical (i.e. normal) stresses act, for the assumption, made on page 46, that the fibers act independently of each other so as not to hinder their horizontal and vertical expansions or contractions, precludes this. This does not, however, mean that horizontal (i.e., longitudinal) shearing stresses cannot occur. It will soon be apparent that these are necessary for the equilibrium of the element.

As the sum of the horizontal forces must equal zero, we have

$$\int\int p\,dy\,dz - \int\int p'dy\,dz + q''\,b\,dx = 0 \quad . \quad (1)$$

where q'' is the horizontal shearing stress on MJ; this is necessary, as $\int\int p'dy\,dz$ is greater than $\int\int p\,dy\,dz$. This becomes evident if we substitute $p = \frac{M_x y}{I}$ and $p' = \frac{(M_x + dM_x)y}{I}$ in equation (1) and obtain

$$q''b\,dx = \int\int \frac{M_x + dM_x}{I} y\,dy\,dz - \int\int \frac{M_x}{I} y\,dy\,dz$$

$$= \int\int \frac{dM_x}{I} y\,dy\,dz,$$

and as M_x and I are constant for the section AB, and the integration involves only the variables y and z, we have

$$q''b\,dx = \frac{dM_x}{I}\int\int y\,dy\,dz,$$

so that

$$q'' = \frac{dM_x}{dx}\,\frac{1}{bI}\int\int y\,dy\,dz = \frac{Q_x}{bI}\int\int y\,dy\,dz,$$

where Q_x is the shearing force at the section AB,
I is the second moment of area of this section,
b is the width of the section,

and $\int\int y\,dy\,dz$ is the first moment of area about the neutral axis of that **part of the section lying above the plane, MN,** at which **horizontal shearing stress q″ acts.**

As the integration of $\int\int y\,dy\,dz$ extends only over the area ME, Fig. 35 (b), and as

$$\bar{y} = \frac{\int\int y\,dy\,dz}{\int\int dy\,dz},$$

where $\bar{y}$ is the distance of the center of area of the area ME from the neutral axis, $\int\int y\,dy\,dz$ can be put equal to y (area of ME).

So that

$$q'' = \frac{Q_x}{bI}\,\bar{y}\sum \Delta A = \frac{Q_x}{bI}\sum y\,(\Delta A),$$

where $\Sigma\,\Delta A$ is the area of ME and $\bar{y}$ is the ordinate of its center of area measured from the neutral axis (see page 50).

EXERCISE 73. Calculate the horizontal shearing stress (a) on a horizontal surface 3 inches above the neutral surface, (b) on the neutral surface, (c) on the extreme fiber, all at a section 2 feet from the left abutment of a uniformly loaded beam 4 inches by 12 inches, span 10 feet, load 1000 pounds per foot run.

Vertical Shear. — To find the magnitude of the vertical shear on any section of a beam, return to Fig. 35. Consider an element dx in length and dy in depth whose lower face coincides with MNJ, Fig. 35 (b). This element is shown as a free body in Fig. 36.

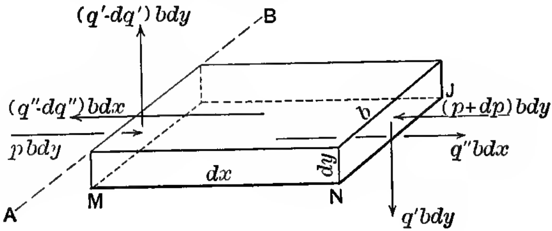


FIG. 36

Taking moments about the axis AB, we have:

$$(q'b\,dy)\,dx - (q''b\,dx)\,dy - (pb\,dy)\frac{dy}{2} + \{(p+dp)\,b\,dy\}\frac{dy}{2} = 0,$$

or
$$q' - q'' + \frac{dp}{2}\left(\frac{dy}{dx}\right) = 0,$$

so that
$$q' = q'',$$

or the **unit vertical shear** q' **equals the unit horizontal shear** q'' **at any point in the material of the beam.**

Hereafter, therefore, we shall use q to represent either the horizontal or vertical unit shear in pounds per square inch.

Variation of the Shear over a Beam Section. Rectangular Section. — As $q = \frac{Q_x}{bI}\bar{y}A$, the shear at any distance y from the neutral axis of a rectangular section (Fig. 37) will be

$$q = \frac{Q_x}{bI}\left\{y + \frac{\frac{h}{2} - y}{2}\right\}\left\{b\left(\frac{h}{2} - y\right)\right\}$$
$$= \frac{Q_x}{I}\left(\frac{h^2}{8} - \frac{y^2}{2}\right) = \frac{3}{2}\frac{Q_x}{bh^3}(h^2 - 4y^2),$$

so that the shear varies as the square of y, as, represented by the parabola to the right of the section in Fig. 37.

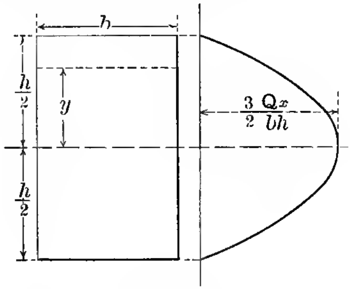


FIG. 37

The maximum value of the shear is

$$q_{max} = \frac{3}{2}\frac{Q_x}{bh},$$

and occurs at the neutral surface.

EXERCISE 74. Prove the above statement.

If the mean shear is defined as $\frac{Q_x}{bh}$; the total shearing force divided by the total area exposed to this shearing force, then the **greatest shear is 50% greater than the mean shear for a rectangular section.**

EXERCISE 75. A beam 4 inches by 6 inches resists a shearing force of 2500 pounds at a certain section. Find the mean shearing stress and the greatest shearing stress on this section.

The dangerous section for shear in a beam thus occurs at the section resisting the greatest shearing force, and the fibers along the neutral axis of this section must withstand the greatest shearing stress; hence these fibers must be investigated for shear.

EXERCISE 76. A simple wooden beam 1 foot long, carrying a central load of 7000 pounds, has a section 1 inch by 12 inches. Calculate the factors of safety for normal stress and shearing stress.

Exercise 77. A wooden cantilever 6 inches long, of rectangular section, is to support a steady, uniformly distributed load of 600 pounds. The width is to be 2 inches. Design the beam for bending and investigate for shear.

EXERCISE 78. Design the beam in Ex. 77 for shear, and investigate for bending.

I-section with Sharp Corners. — A standard 24-inch I-beam, when the slope of the flanges is neglected and an average depth of flange is used, has approximately the dimensions shown in Fig. 38.

The shear at A, the lower edge of the upper flange, is

$$q_a = \frac{Q_x}{Ib}\sum y(\Delta A) = \frac{Q_x}{I\,(7)}(11.55)\,(7 \times .9) = \frac{Q_x}{I}\,(10.4),$$

at B, the upper edge of the web,

$$q_b = \frac{Q_x}{I(.5)}(11.55)\,(7 \times .9) = \frac{Q_x}{I}\,(145.6),$$

at C, the neutral axis,

$$q_c = \frac{Q_x}{Ib}\sum y(\Delta A) = \frac{Q_x}{I(.5)}\{(7\times .9)(11.55)+(.5\times 11.1)(5.5)\}$$

$$= \frac{Q_x}{I}(207.2).$$

As the variation in the shear follows the parabolic law in the interval from the extreme fiber to A, also from B

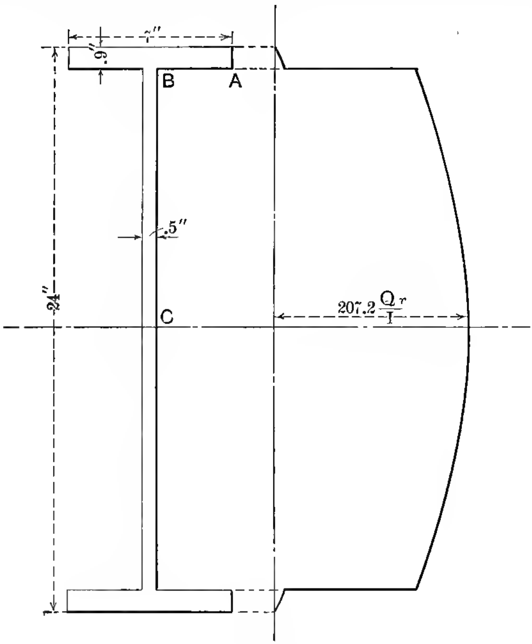


FIG. 38

to C, the variation may be represented by the curve to the right of Fig. 38.

This graph clearly shows that in beams of I-section the material of the web may for practical calculations be assumed to resist the whole shearing force. As already noted, the normal stresses due to bending reach their greatest values at the extreme fibers of the beam, therefore in practice the material in the flanges alone may be considered to resist the bending moment. These conclusions also hold in the case of built-up girders having approximately **I**-shaped sections.

The **total shearing force divided by the area of the web** is under these assumptions regarded as the unit shear throughout the web.

The distribution of material in these sections is evidently best suited to the case in hand, the material being placed where it will be stressed approximately to its working stress throughout.

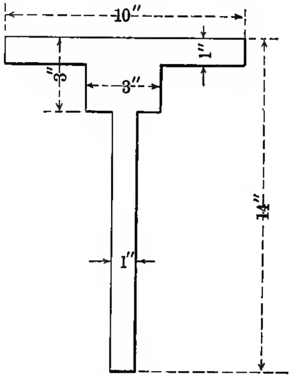


FIG. 39

EXERCISE 79. Calculate the shear for the principal points in the section shown in Fig. 39. The section approximates the section of the beam shown on page 51.

Circular Sections. — The variation of the shear over circular sections is more difficult to determine, for now the free surface of the beam is no longer vertical or horizontal, as in the case of rectangular sections.

The shear at A (Fig. 40) cannot be vertical, for if it were this would entail a component shear along the radius and normal to the free surface at A, and this in turn would entail an equal shear in the tangent plane at

A and parallel to the axis of the beam (see page 134). This last shear **in the tangent plane at A** cannot exist without an external force acting in this same plane or at right angles to it, and such forces are excluded in our assumed conditions of loading.

Thus the only shear which can exist at A is a tangential shear having the direction AO or OA and indicated by the arrow q.

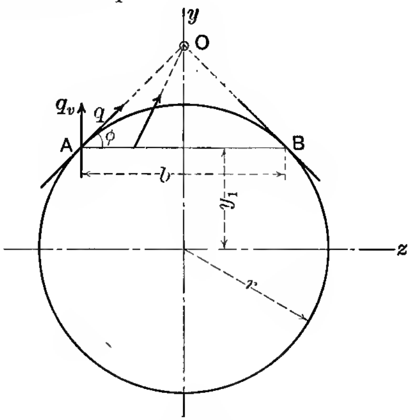


FIG. 40

In Fig. 40 consider the shear at a distance y_1 from the neutral axis. At the point A the shear must take the direction of the tangent to the circle, for we assume no longitudinal forces at this surface. The shearing stresses at other points along the line AB are assumed to pass through the point O for want of a better guide as to their directions. The vertical component of the shear must still follow the law $q = \frac{Q_x}{Ib}\int\int y\,dy\,dz$, where

$$\int\int y\,dy\,dz = 2\int_{y_1}^{r} y\sqrt{r^2 - y^2}\,dy = \frac{2}{3}\sqrt{(r^2 - y_1^2)^3} = \frac{b^3}{12};$$

and b is the width of the section at a distance y_1 from the neutral axis, so that the vertical component of all shears along AB is,

$$q_v = \frac{Q_x b^2}{I(12)} = \frac{Q_x b^2}{3\pi r^4}.$$

At A the total shear q can be found from

$$\frac{q}{q_v} = \frac{r}{\frac{b}{2}} \quad \text{or} \quad q = \frac{2r}{b} q_v,$$

whence,
$$q = \frac{2 Q_x b}{3\pi r^3}.$$

The maximum value of q occurs at the neutral axis, where $b = 2r$; thus

$$q_{max} = \frac{4 Q_x}{3\pi r^2} = \frac{4}{3}\text{(mean unit shear)}.$$

It should be noted that this calculation does not apply rigorously to rivets, for here external forces due to friction act along the circumference of the rivet.

In general, it follows from the above that the shear at and tangent to the boundary of any section symmetrical about a vertical axis is

$$q = \frac{Q_x}{bI\cos\phi}\bar{y}\,\Sigma\Delta A,$$

where ϕ is the angle between the tangent to the boundary at the end of the chord considered and the chord itself, as indicated in Fig. 40.

Criteria for Equal Strength in Shear and Bending for Beams of Rectangular Section. — Consider, as an illustration of the method of obtaining these criteria, the case of a simple beam loaded at mid-span (Fig. 41) and rectangular in section.

Here the greatest stress due to bending is

$$p' = \frac{My}{I} = \frac{Wl}{4} \cdot \frac{h}{2} \cdot \frac{12}{bh^3} = \frac{3Wl}{2\,bh^2},$$

and for the greatest shearing stress we have

$$q' = \frac{Q_x}{Ib}\bar{y}A = \frac{W}{2} \cdot \frac{12}{bh^3} \cdot \frac{1}{b} \cdot \frac{h}{4} \cdot \frac{bh}{2} = \frac{3}{4}\frac{W}{hb}.$$

Thus

$$\frac{\text{working normal stress}}{\text{working shearing stress}} = \frac{p'}{q'} = \frac{3\,Wl}{2\,bh^2} \cdot \frac{4\,bh}{3\,W} = \frac{2\,l}{h},$$

or if twice the span divided by the depth of the beam is equal to or less than the ratio of the unit normal stress

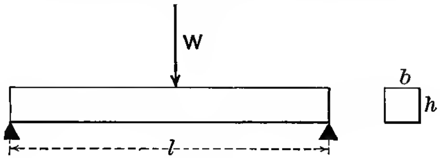


FIG. 41.

to the unit shearing stress of the material, the beam should be designed for shear.

EXERCISE 80. What is the ratio of the length to the depth of a simple rectangular beam loaded at mid-span equally strong in shear and bending if the material is (a) wood, (b) steel?

EXERCISE 81. Deduce the criterion for a uniformly loaded simple beam.

EXERCISE 82. Same as Ex. 81, for a cantilever loaded at its end.

EXERCISE 83. Same as Ex. 81, for a uniformly loaded cantilever beam.

CHAPTER III

DEFLECTION OF BEAMS DUE TO SIMPLE BENDING

SECTION VII

THE DIFFERENTIAL EQUATION OF THE ELASTIC CURVE AND ITS APPLICATION

IN the previous chapter the strength of beams has been considered. It often happens that strength is not the only factor to consider in the design of beams. The deformation of the beam due to a load must sometimes be taken into account; thus the **stiffness as well as the strength** of beams must often be investigated.

On page 48, the relation between the bending moment at any section and the radius of curvature of the elastic curve (see page 23) of the beam has been discussed. This relation is expressed by the formula

$$M = \frac{E}{R} I,$$

or

$$\frac{1}{R} = \frac{M}{EI}.$$

The differential equation of the elastic curve can now be obtained by placing

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

the well-known expression for the radius of curvature. In practical problems the slope, $\left(\frac{dy}{dx}\right)$, of the beam is always very small, so that $\left(\frac{dy}{dx}\right)^2$ is negligible when added to unity, and the simplified value for the radius of curvature is

$$R = \frac{1}{\frac{d^2y}{dx^2}}, \quad \text{approximately.}$$

The **differential equation of the elastic curve** is thus

$$\mathbf{EI\frac{d^2y}{dx^2} = M_x.}$$

In this equation x and y represent the coördinates of any point on the elastic curve, referred to the intersection of the neutral surface of the unstressed beam and the plane of bending as x-axis, and the line normal to this axis through the left end of the beam as y-axis; M_x is the bending moment and I the second moment of area of the beam section, both about the neutral axis of the section which passes through the point (x, y). E, of course, is Young's Modulus.

As an **application** to a concrete problem, let us **find the slope at any point, the greatest slope, the equation of the elastic curve, and the greatest deflection** of a cantilever loaded at its end only (Fig. 42).

In the figure the neutral surface before and after loading is shown. At any section AB at a distance x from the left end of the beam the deflection is y and the bending moment (paying due regard to the sign) is $-\;[W(l - x)]$, so that

$$EI\frac{d^2y}{dx^2} = -\,W(l - x). \quad . \quad . \quad . \quad . \quad (1)$$

Note that for all values of x between 0 and l the $\frac{d^2y}{dx^2}$ will be negative; this shows that the **elastic curve is concave downward,** as experience would dictate.

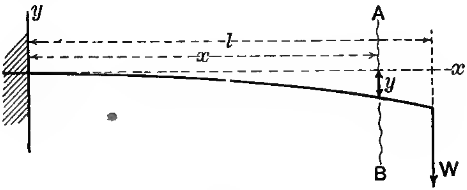


FIG. 42.

Integrating the differential equation twice and introducing the constants of integration, we obtain

$$EI\left(\frac{dy}{dx}\right) = + \frac{W}{2}(l - x)^2 + C_1, \quad . \quad . \quad . \quad (2)$$

$$EI\,y = - \frac{W}{6}(l - x)^3 + C_1x + C_2, \quad . \quad (3)$$

here $\frac{dy}{dx}$ represents the slope of the elastic curve at any point, and y the deflection at this point.

To find the constants of integration the end conditions must be satisfied. Thus, from the manner in which the beam is supported, as illustrated in Fig. 42, $\frac{dy}{dx} = 0$, when $x = 0$, and also $y = 0$ when $x = 0$.

From the first of these conditions C_1 can be obtained from equation (2)
thus,

$$EI(0) = \frac{W}{2}(l - 0)^2 + C_1 \quad \text{or} \quad C_1 = -\frac{W}{2}l^2.$$

Similarly, from equation (3) by reason of the second condition we have

$$EI(0) = -\frac{W}{6}(l-0)^3 - \frac{W}{2}l^2(0) + C_2 \text{ or } C_2 = \frac{W}{6}l^3.$$

Hence, the slope at any point is

$$\frac{dy}{dx} = \frac{W}{2EI}\{(l-x)^2 - l^2\} = \frac{W}{2EI}\{x^2 - 2lx\}, \quad (4)$$

and the deflection at any point is

$$y = -\frac{W}{6EI}\{(l-x)^3 + 3l^2x - l^3\} = -\frac{W}{6EI}\{3lx^2 - x^3\}. \quad (5)$$

The greatest slope occurs when $x = l$, so that from equation (4)

$$\left(\frac{dy}{dx}\right)_{\text{greatest}} = -\frac{Wl^2}{2EI}, \quad . \quad . \quad . \quad . \quad (6)$$

and the greatest deflection at the free end, as obtained from equation (5) by placing $x = l$, is

$$y_{\text{greatest}} = -\frac{Wl^3}{3EI}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

From the last result the **greatest deflection for a given greatest fiber stress** can readily be found by the use of the formula $M = \frac{pI}{y}$. The greatest fiber stress, p_c, occurs at the extreme fiber at the wall (the dangerous section); here $M = Wl$, so that $Wl = \frac{p_cI}{c}$ where c is the distance from the neutral axis to the extreme fiber, and $W = \frac{p_cI}{lc}$, which on substitution in equation (7) yields instead of y_{greatest},

$$y_p = -\frac{p_cl^2}{3Ec},$$

the deflection for which the fiber stress at the extreme fiber in the dangerous section is p_c.

Exercise 84. Solve the above problem when the cantilever is fixed at the right end and the origin is assumed at the left end.

Exercise 85. A wooden cantilever is 5 feet long and 4 inches by 8 inches in section, the longest side being vertical. The concentrated load at the end is 800 pounds and $E =$ 1,400,000 pounds per square inch. Calculate the greatest slope and deflection.

Exercise 86. (a) What will be the greatest allowable deflection for the beam described in Ex. 85, if the fiber stress (i.e., the stress on any fiber) is not to exceed 1200 pounds per square inch?

(b) Will the load under these conditions be 800 pounds?

Exercise 87. Find the equation of the elastic curve, the greatest slope, the greatest deflection, and the greatest deflection for a given fiber stress for a uniformly loaded cantilever l inches long and loaded with w pounds per inch.

Exercise 88. Same as Ex. 87, for a simple beam, span l inches, and loaded with w pounds per inch. Origin at left support.

Exercise 89. Determine the proper distance from center to center for 12-inch steel I-beams ($I = 215.8$ inches^4), span 24 feet, to support a uniform load of 100 pounds per square foot of floor area with a maximum deflection of $\frac{1}{360}$ of the span.

Will these beams be strong enough to bear the load?

Exercise 90. What is the greatest span that a 10-inch steel I-beam ($I = 122.1$ inches^4), supported at both ends and uniformly loaded, can bridge, if the greatest fiber stress is not to exceed 16,000 pounds per square inch, and the maximum deflection is to be $\frac{1}{360}$ of the span? What total load will this beam bear?

EXERCISE 91. Find the equation of the elastic curve, the greatest slope, the maximum deflection, and the deflection which will produce a certain greatest fiber stress, p_c, for a simple beam, span l, loaded at mid-span with W pounds. (Note that in this case there are two intervals to consider; find the equation of the elastic curve for each interval, using the left abutment as origin for both intervals.) In this problem is the greatest deflection a maximum deflection?

In problems involving two intervals, such as Ex. 91, it is advisable to use a different origin for each interval. This artifice is solely used to simplify the mathematical calculations. Great care must be taken to keep this change of origin constantly in mind during the calculation and in the application of the resulting equations of the elastic curve.

To illustrate, consider a **simple beam excentrically loaded** (Fig. 43). Here for the left interval assume the

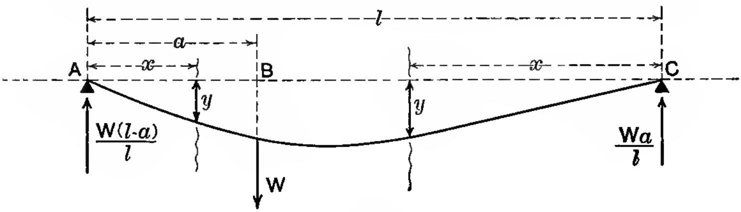


FIG. 43

origin at A, x increasing positively toward B; for the right interval let C be the origin and assume x to increase positively towards B. Then for the left interval we have

$$EI\frac{d^2y}{dx^2} = \frac{W(l-a)}{l}x, \qquad (1)$$

$$EI\frac{dy}{dx} = \frac{W(l-a)}{2l}x^2 + C_1, \qquad (2)$$

and $$EIy = \frac{W(l-a)}{6l}x^3 + C_1x + C_2. \quad . \quad . \quad (3)$$

For the right interval

$$EI\frac{d^2y}{dx^2} = \frac{Wa}{l}x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$EI\frac{dy}{dx} = \frac{Wa}{2l}x^2 + C_3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and $$EIy = \frac{Wa}{6l}x^3 + C_3x + C_4. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the left interval when $x = 0$, $y = 0$; so that $C_2 = 0$. In the right interval when $x = 0$, $y = 0$; so that $C_4 = 0$.

The evaluation of C_1 and C_3 is more difficult. It should be noted that the deflections and the slopes at B, the meeting point of the intervals, must be the same for both parts of the elastic curve; for the beam forms a continuous curve, although the equation of its parts are different.

Thus the slope at B from (2) [for $x = a$] must equal the slope at B from (5) [for $x = l - a$], provided we change the sign of the latter; for these slopes are numerically equal but opposite in sign by reason of our assumption of the positive direction from C to B in the right interval

or $$\frac{W(l-a)}{2l}a^2 + C_1 = -\left[\frac{Wa}{2l}(l-a)^2 + C_3\right]. \quad (7)$$

Similarly, from (3) [for $x = a$] and (6) [for $x = l - a$] we have

$$\frac{W(l-a)}{6l}a^3 + C_1a = \frac{Wa}{6l}(l-a)^3 + C_3(l-a). \quad (8)$$

Equation (7) gives $$C_1 + C_3 = -\frac{Wa(l-a)l}{2l}.$$

Equation (8) gives $C_1a - C_3(l-a) = \dfrac{Wa(l-a)l}{6l}(l-2a)$.

By elimination C_1 and C_3 may be found to be

$$C_1 = \frac{Wa(l-a)(a-2l)}{6l},$$

$$C_3 = \frac{Wa(l-a)(-l-a)}{6l}.$$

The equations of the elastic curve are, then, for the left with origin at A

$$y = \frac{W(l-a)x}{6lEI}(x^2 + a^2 - 2al), \quad . \quad . \quad (9)$$

for the right with origin at C

$$y = \frac{Wax}{6lEI}(x^2 + a^2 - l^2). \quad . \quad . \quad . \quad (10)$$

To find the **deflection under the load,** put $x = a$ in equation (9)

whence, $$y_B = -\frac{W(l-a)^2a^2}{3lEI};$$

or put $x = l - a$ in equation (10)

and again, $$y_B = -\frac{Wa^2(l-a)^2}{3lEI}.$$

To find the **maximum deflection** the corresponding value of x must be found.

As the maximum deflection evidently occurs in the longer interval, and as we have tacitly assumed $a < \dfrac{l}{2}$ put the first derivative of equation (10) equal to zero,

whence, $$\frac{Wa}{6lEI}(3x^2 + a^2 - l^2) = 0,$$

whence $x = \sqrt{\frac{l^2 - a^2}{3}}$, the distance from C at which the maximum deflection occurs.

This value of x substituted in equation (10) gives

$$y_{\max} = -\frac{Wa\,(l^2 - a^2)}{9\,lEI}\sqrt{\frac{l^2 - a^2}{3}}.$$

EXERCISE 92. What result is obtained for $y_{\max}$ from equation (9)? Interpret your answer.

EXERCISE 93. (a) Find the deflection at the quarter points (for which $x = \frac{l}{4}$ in either interval) for a simple beam loaded at one quarter point.

(b) What is the maximum deflection and its location under the loading in (a)?

EXERCISE 94. From equations (1) to (10) of the above example obtain the answers to Ex. 91.

When a **simple beam supports two loads symmetrically placed** (Fig. 44), it is much easier to find the deflections by the **principle of superposition** and the above results, than by a new integration.

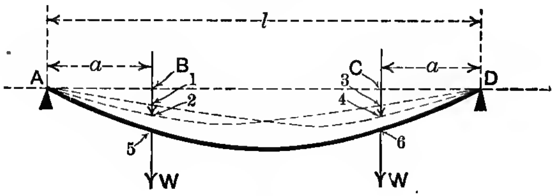


FIG. 44

In Fig. 44 the dotted lines show the deflections due to each load acting separately and the full line shows the deflections due to both loads acting simultaneously.

These latter deflections are evidently the sum of the corresponding deflections due to each load acting alone.

By reason of the symmetrical loading $B\,2 = C\,4$ and $B\,1 = C\,3$; by the principle of superposition $B\,5 = B\,2 + B\,1$, whence $B\,5 = B\,2 + C\,3$.

From equation (9), page 82, for $x = a$

$$B\,2 = -\frac{W(l-a)^2a^2}{3\,lEI},$$

and from equation (10), page 82, for $x = a$,

$$(C\,3) = -\frac{Wa^2(l^2 - 2\,a^2)}{6\,lEI},$$

so that the required deflection

$$B\,5 = (B\,1) + (C\,3) = -\frac{Wa^2}{6\,lEI}(2\,l^2 - 4\,al + 2\,a^2 + l^2 - 2a^2),$$

$$= -\frac{Wa^2}{6\,EI}(3\,l - 4\,a).$$

Exercise 95. Find the deflection under either load if the beam is loaded at the quarter points.

Exercise 96. Find the maximum deflection for the beam shown in Fig. 44.

The two following exercises should be solved, without calculus, by using any appropriate equations deduced in this chapter.

Exercise 97. A simple beam l inches long rests on end supports and bears a total uniform load of W pounds. Another support just touches the bent beam at mid-span. How much must this middle support be raised in order that the end supports shall just touch the beam but support no load?

Exercise 98. A wooden cantilever 15 feet long, 3 inches wide, and 4 inches deep carries a load of 100 pounds 5 feet from the free end. Find the deflection at the end due to this load.

Propped Beams. — Simple beams supported at another point besides the ends, or cantilevers supported at some point other than the fixed end, are called propped beams. The pressure upon these extra props cannot be found by means of the principles of statics alone; the elastic properties of the beam must be considered in their determination. The following example will illustrate the application of the principle of superposition to such problems.

Consider a uniformly loaded cantilever supported by a prop at its end, this prop to maintain the free end on a level with the fixed end of the beam (Fig. 45).

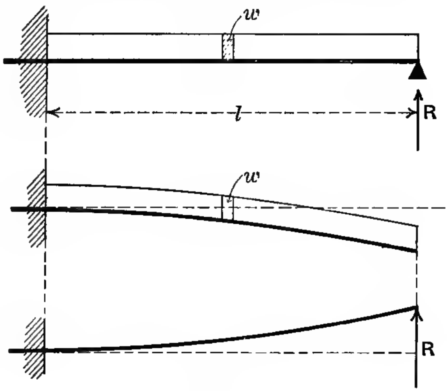


FIG. 45

If the prop is removed the deflection at the end will be $\frac{wl^4}{8\,EI}$ (see Ex. 87). Now the reaction of the prop, R, must equal the concentrated force, acting alone, necessary to deflect the end of the cantilever through $\frac{wl^4}{8\,EI}$ inches so that it returns to the required level. Thus from the formula $y_{\text{greatest}} = \frac{Wl^3}{3\,EI}$ (page 78)

we have $\frac{wl^4}{8\,EI} = \frac{Rl^3}{3\,EI}$ or $R = \frac{3\,wl}{8}$ pounds,

the pressure on the prop.

Exercise 99. Find the pressure on a prop which keeps the free end of a cantilever (loaded at the center of its length, l, with a concentrated load, W) at the same level as the fixed end.

Exercise 100. A beam is continuous over three supports, each span is l inches long and the load is w pounds per inch run. Find the pressure on each of the three props, (a) by considering the problem as a simple beam; (b) by considering the problem as a cantilever.

Exercise 101. A beam is continuous over three supports, each span l inches long; the only loads are W pounds concentrated at each mid-span. Find the reactions of supports.

Exercise 102. A cantilever, length, l, carries a uniformly distributed load, w, over three-fourths of its length from the fixed end, and is propped at the free end to the level of the fixed end. What force acts upon the prop?

Section VIII

THE DIFFERENTIAL EQUATIONS OF BEAMS

On page 33 it was shown that

$$w_x = \frac{dQ_x}{dx},$$

and that

$$Q_x = \frac{dM_x}{dx};$$

followed on page 76 by the proof that

$$\frac{M}{EI} = \frac{d^2y}{dx^2}.$$

Combining these equations, the space rate of loading, w_x, can be expressed as follows:

$$Q_x = \frac{dM_x}{dx} = EI\frac{d^3y}{dx^3},$$

and $$w_x = \frac{dQ_x}{dx} = EI\frac{d^4y}{dx^4}.$$

As an illustration of the integration of this equation of the fourth order, consider the deflection of a simple beam loaded with a distributed load varying gradually from zero, over one abutment, to w pounds per inch over the other abutment.

Fig. 46 illustrates the loading. In this case $w_x = \frac{wx}{l}$, so that

$$EI\frac{d^4y}{dx^4} = -\frac{wx}{l},$$

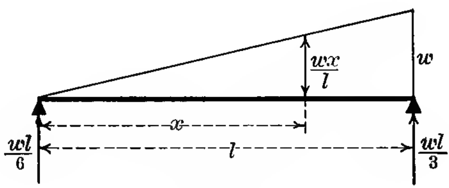


FIG. 46

the minus sign indicating downward loading. Successive integrations give

$$EI\frac{d^3y}{dx^3} = -\frac{wx^2}{2l} + C_1, \quad \quad (1)$$

$$EI\frac{d^2y}{dx^2} = -\frac{wx^3}{6l} + C_1x + C_2, \quad \quad (2)$$

$$EI\frac{dy}{dx} = -\frac{wx^4}{24\,l} + \frac{C_1x^2}{2} + C_2x + C_3, \quad . \quad . \quad . \quad (3)$$

$$EIy = -\frac{wx^5}{120\,l} + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4. \quad . \quad (4)$$

Now as $EI\frac{d^3y}{dx^3}$ represents the shearing force Q_x, at the point x, C_1 can be found by putting $EI\frac{d^3y}{dx^3} = \frac{wl}{6}$ when $x = 0$, for the shearing force at the left end equals the reaction at that point; hence

$$C_1 = +\frac{wl}{6}.$$

Again, $EI\frac{d^2y}{dx^2}$ is the bending moment at any section x units from the left origin, and as the bending moment

$$EI\frac{d^2y}{dx^2} = 0 \text{ when } x = 0,$$

$$C_2 = 0.$$

As the slope is not known at any point of the beam, pass to equation (4).

Here $y = 0$ when $x = 0$, $\therefore C_4 = 0$.

Also $y = 0$ when $x = l$,

so that $$C_3l = +\frac{wl^4}{120} - \frac{wl^4}{36} - 0 = -\frac{7\,wl^4}{360},$$

or $$C_3 = -\frac{7\,wl^3}{360}.$$

Substituting the values of the constants of integration just found in equations (1), (2), (3), and (4), we obtain

$$w_x = EI\frac{d^4y}{dx^4} = -\frac{wx}{l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$Q_x = EI\frac{d^3y}{dx^3} = -\frac{w}{6l}\{3x^2 - l^2\} \quad . \quad . \quad . \quad . \quad (6)$$

$$M_x = EI\frac{d^2y}{dx^2} = -\frac{w}{6l}\{x^3 - l^2x\}, \quad . \quad . \quad . \quad . \quad (7)$$

$$\text{slope} = \frac{dy}{dx} = -\frac{w}{360\,EIl}\{15x^4 - 30\,l^2x^2 + 7\,l^4\}, \quad (8)$$

$$\text{deflection} = y = -\frac{w}{360\,EIl}\{3x^5 - 10\,l^2x^3 + 7\,l^4x\}. \quad (9)$$

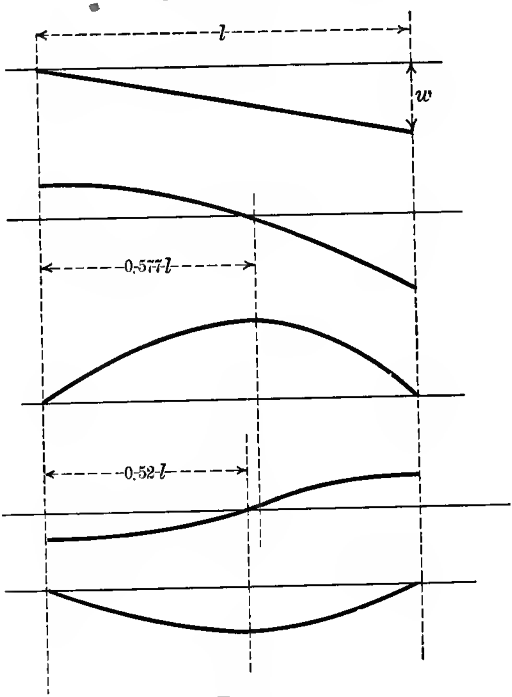


FIG. 47

From equations (8) and (9) the slopes and deflections including the greatest and maximum values can be obtained as before.

In Fig. 47 the equations (5) to (9) are plotted; note their relations as derivative curves, and compute all principal values and their location and indicate same on Fig. 47.

EXERCISE 103. Deduce the above results from the equation $M_x = EI \frac{d^2y}{dx^2}$.

CHAPTER IV

STATICALLY INDETERMINATE BEAMS

Section IX

PROPPED AND BUILT-IN BEAMS

Whenever a free body is in equilibrium under the action of coplanar forces, three independent equations, and only three, can be written expressive of the conditions of equilibrium. Thus only three unknown quantities can in general exist in such a problem.

Any problem in planar equilibrium involving more than three unknown quantities cannot be solved by the principles of statics alone, but requires in addition the use of the principles of mechanics of materials; such problems are said to be **statically indeterminate.**

In Fig. 48 (a) the reactions at the wall involve an unknown force, R_1, and an unknown couple, M_1, and in addition there is the unknown force R at the propped end. As the forces in this problem are parallel, statics furnishes only two equations, and thus the problem is statically indeterminate.

To illustrate the method of procedure in problems of this sort, consider this specific problem.

A horizontal beam, fixed at one end and supported at the other end at the same level as the fixed end, carries a uniformly distributed load; it is required to find (1) the reaction of the support, (2) the equation of the elastic curve, (3) the dangerous section, (4) the deflections, (5) the point of inflection, and (6) to sketch the diagrams of shearing forces, bending movements, slopes, and deflections.

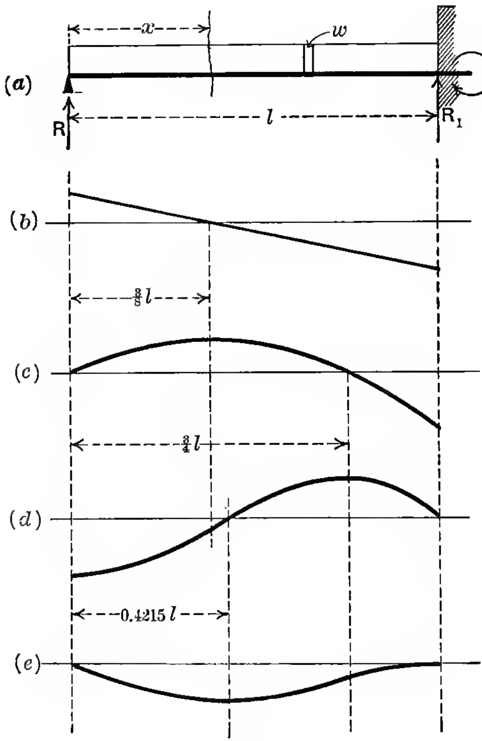


FIG. 48

Fig. 48 (a) illustrates the problem. It is impossible to find the reaction R by the statical conditions of equilibrium.

The start is thus made with the differential equation of the bent beam.

$$EI\frac{d^2y}{dx^2} = M_x = Rx - \frac{wx^2}{2}, \quad . \quad . \quad . \quad (1)$$

whence
$$EI\left(\frac{dy}{dx}\right) = R\frac{x^2}{2} - \frac{wx^3}{6} + C_1, \quad . \quad . \quad . \quad (2)$$

and
$$EIy = R\frac{x^3}{6} - \frac{wx^4}{24} + C_1x + C_2. \quad . \quad (3)$$

It now remains to find the constants of integration C_1 and C_2, as well as the reaction R. To accomplish this we have the following conditions:

$$x = l \text{ when } \left(\frac{dy}{dx}\right) = 0, \quad . \quad . \quad . \quad . \quad . \quad \text{I}$$

$$x = 0 \text{ when } y = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{II}$$

$$x = l \text{ when } y = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{III}$$

From condition I it follows that $C_1 = -\frac{Rl^2}{2} + \frac{wl^3}{6}$.

From condition II it follows that $C_2 = 0$.
From condition III we have

$$\frac{Rl^3}{6} - \frac{wl^4}{24} - \frac{Rl^3}{2} + \frac{wl^4}{6} = 0,$$

whence
$$R = \frac{3}{8}wl,$$

and thus
$$C_1 = -\frac{wl^3}{48}.$$

Equations (1), (2), and (3) can now be rewritten as follows:

$$M_x = EI\left(\frac{d^2y}{dx^2}\right) = \frac{wx}{8}(3\,l - 4\,x), \quad . \quad . \quad (4)$$

$$EI\left(\frac{dy}{dx}\right) = \frac{w}{48}(9\,lx^2 - 8\,x^3 - l^3), \quad . \quad . \quad . \quad (5)$$

$$EIy = \frac{wx}{48}(3\,lx^2 - 2\,x^3 - l^3). \quad . \quad . \quad (6)$$

To locate the dangerous section, put

$$Q_x = \frac{dM_x}{dx} = \frac{w}{8}(3\,l - 8\,x) = 0,$$

whence
$$x = \frac{3\,l}{8},$$

and the maximum bending moment is $+\frac{9}{128}wl^2$, but the bending moment at the wall is $-\frac{wl^2}{8}$, therefore the **dangerous section is at the wall.**

The **point of inflection** of the elastic curve occurs when $\frac{d^2y}{dx^2} = 0$, as is demonstrated in calculus, so that equation (4) equated to zero gives $x = 0$ and $\frac{3\,l}{4}$. The point of inflection is located at three-fourths of the span from the propped end. Interpret $x = 0$.

The maximum deflection occurs when the slope is zero. Equating (5) to zero and solving for x, we find

$$x = .4215\,l \quad \text{and} \quad y_{\max} = -\ .0054\,\frac{wl^4}{EI}\cdot$$

All these results can best be summed up in diagrams of S.F., B.M., slopes, and deflections shown in Fig. 48 (b), (c), (d), and (e) respectively.

EXERCISE 104. Same conditions and requirements as in the above example, except that the beam is weightless and loaded at mid-span with W pounds.

As another illustration, consider **a weightless beam built in at both ends and loaded at mid-span with W pounds.** (Fig. 49 (a)).

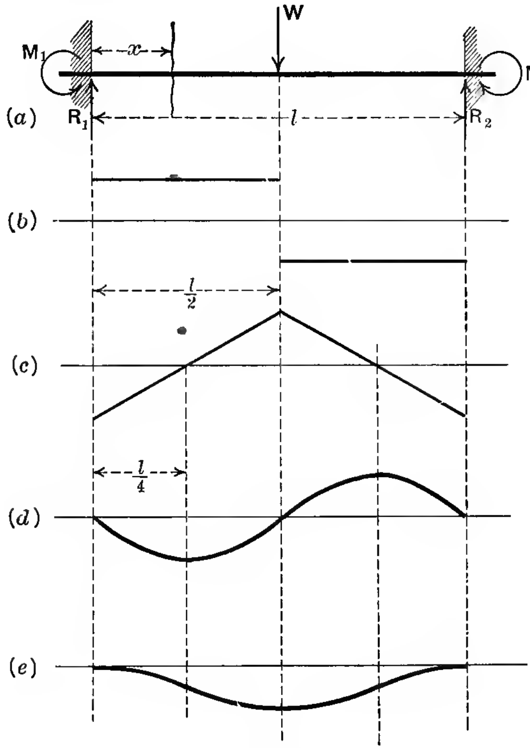


FIG. 49

Here each wall must support one-half of the load, for the sum of the vertical forces must be zero and by symmetry the reactions must be equal; thus $R_1 = R_2 = \frac{W}{2}$. Moreover, as the ends of the beam are kept horizontal, equal moments M_1 and M_2 must act at the ends.

Thus, for the interval $x = 0$ to $x = \frac{l}{2}$,

$$EI\frac{d^2y}{dx^2} = M_x = \frac{W}{2}x - M_1,$$

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} - M_1x + C_1,$$

and $$EIy = \frac{Wx^3}{12} - M_1\frac{x^2}{2} + C_1x + C_2.$$

As $\frac{dy}{dx} = 0$ when $x = 0$, $C_1 = 0$;

and as $y = 0$ when $x = 0$, $C_2 = 0$.

Also by reason of the symmetrical loading

$$\frac{dy}{dx} = 0 \quad \text{when} \quad x = \frac{l}{2},$$

whence $$M_1 = \frac{Wl}{8}.$$

So that $$EI\frac{d^2y}{dx^2} = \frac{W}{8}(4x - l),$$

$$EI\frac{dy}{dx} = \frac{Wx}{8}(2x - l),$$

and $$EIy = \frac{Wx^2}{48}(4x - 3l).$$

The greatest B.M. is at mid-span and equals $\frac{Wl}{8}$, or at the wall, where the B.M. is $-\frac{Wl}{8}$.

The greatest deflection is $-\frac{Wl^3}{192\,EI}$ at mid-span, and the points of inflection are at $x = \frac{l}{4}$ and $\frac{3l}{4}$.

The curves of shearing forces, bending moments, slopes, and deflections are shown in Fig. 49 (b), (c), (d), and (e), respectively. It should be noted that there are two intervals and that the equations of these curves for the right-hand interval have not been found.

EXERCISE 105. Find the reaction, bending moment at the wall, maximum bending moment, dangerous section, maximum deflection, points of inflection, and deflection at the center of the span for a given fiber stress for a uniformly loaded beam "fixed" at both ends.

SECTION X

CONTINUOUS BEAMS

Beams extending without break over more than two supports are known as continuous beams.

It is impossible to find the reactions of the supports of a continuous beam by the principle of statics alone, so that here again we meet a statically indeterminate problem.

EXERCISE 106. Show by the methods of the previous section that the reactions of the two center supports of a continuous beam extending over three equal spans, l, and uniformly loaded are each $\frac{11}{10}wl$. Assume the origin for the first span at the left-hand abutment and for the next span at its left-hand abutment.

Clapeyron's Theorem of Three Moments. — The calculation of the reactions of the supports of a continuous beam by the methods outlined above is a tedious operation, as Ex. 106 shows.

The theorem of three moments gathers in a single statement the results of the calculations to be performed and obviates their continuous repetition.

The proof of this theorem, when the spans are uniformly loaded and the supports are all at the same level, follows.

Consider any two adjacent spans of a continuous beam (Fig. 50), the loading of each span to be uniformly

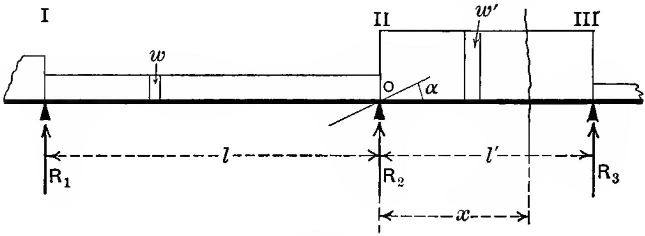


FIG. 50

distributed but not necessarily of equal intensity for both spans.

Let M_1, M_2, M_3 be the bending moments at the supports (I), (II), and (III) respectively, and R_1, R_2, R_3 the corresponding reactions.

Consider span (II)-(III) with origin at (II)

then $$EI\left(\frac{d^4y}{dx^4}\right) = -w', \quad \ldots \quad (1)$$

whence $$EI\left(\frac{d^3y}{dx^3}\right) = -w'x + C_1, \quad \ldots \quad (2)$$

and $$EI\left(\frac{d^2y}{dx^2}\right) = -\frac{w'x^2}{2} + C_1x + C_2, \quad \ldots \quad (3)$$

and as $$EI\left(\frac{d^2y}{dx^2}\right) = M_x,$$

the bending moment at any section x in the span (II)-(III) is

$$M_x = -\frac{w'x^2}{2} + C_1x + C_2.$$

To find C_1, note that $EI\left(\frac{d^3y}{dx^3}\right)$ in equation (2) equals Q_x, the shear at the section at which the bending moment is M_x, and as x diminishes Q_x approaches more and more the value of the shear infinitely close to (II) but on its right. (Study Fig. 52 (a), page 104, where the full lines indicate the shearing forces acting in the continuous beam.)

Thus, if F_2' represents the shear infinitely close to (II) but on its right, then $Q_x = F_2'$ when $x = 0$ and by equation (2)

$$C_1 = F_2',$$

so that $$M_x = -\frac{w'x^2}{2} + F_2'x + C_2;$$

also when $x = 0, \quad M_x = M_2, \quad \therefore C_2 = M_2,$

and we have,

$$M_x = -\frac{w'x^2}{2} + F_2'x + M_2. \quad . \quad . \quad . \quad . \quad (4)$$

Place

$$EI\left(\frac{d^2y}{dx^2}\right) = M_x = -\frac{w'x^2}{2} + F_2'x + M_2, \quad . \quad . \quad (5)$$

and integrate, then

$$EI\left(\frac{dy}{dx}\right) = -\frac{w'x^3}{6} + \frac{F_2'x^2}{2} + M_2x + C_3. \quad . \quad (6)$$

If we designate the slope of the elastic curve at (II) by $\tan \alpha$, we may put $x = 0$ and $\frac{dy}{dx} = \tan \alpha$, so that $$C_3 = EI\,(\tan\alpha),$$

and

$$EI\left(\frac{dy}{dx} - \tan\alpha\right) = -\frac{w'x^3}{6} + \frac{F_2'x^2}{2} + M_2x. \quad (7)$$

Integrating once more

$$EI\,(y - x\tan\alpha) = -\frac{w'x^4}{24} + \frac{F_2'x^3}{6} + \frac{M_2x^2}{2}$$

$$+ [C = 0 \text{ as } x = 0 \text{ when } y = 0]. \quad (8)$$

Also as $y = 0$ when $x = l'$ from (8) we have

$$-EI\tan\alpha = -\frac{w'l'^3}{24} + \frac{F_2'l'^2}{6} + \frac{M_2l'}{2}, \quad (9)$$

from equation (4) when $x = l'$,

we obtain $$M_3 = -\frac{w'l'^2}{2} + F_2'l' + M_2, \quad (10)$$

from which F_2' may be found; this when substituted in equation (9) gives

$$-EI\tan\alpha = \frac{w'l'^3}{24} + \frac{M_2l'}{3} + \frac{M_3l'}{6}. \quad (11)$$

From span (I)-(II), using (II) as origin so that x is **positive** when measured towards the **left,** we obtain

$$+EI\tan\alpha = \frac{wl^3}{24} + \frac{M_2l}{3} + \frac{M_1l}{6}. \quad (12)$$

Adding equations (11) and (12), we have

$$M_1l + 2\,M_2(l + l') + M_3\,l' = -\frac{wl^3 + w'l'^3}{4},$$

this is the theorem of three moments. It furnishes a relation between the bending moments at three consecutive supports and the loading. By applying this theorem to pairs of successive, consecutive spans, as many equations as there are supports less two can be obtained and the end conditions will supply the two remaining equations necessary for finding the bending moment over each support (see page 102).

The M's being known, the F's can be calculated from equations similar to equation (10).

The points of maximum moments are of course found by placing $\frac{dM_x}{dx} = Q_x = 0$, and the points of inflection by putting $M_x = 0$.

The reaction may be found by adding the values of the F's infinitely close to the support and on either side of it. (See Fig. 52 (a), page 104.)

For instance, the reaction at (II) is $R_2 = F_2' + F_2$ if F_2 is the shear in span (I)-(II) infinitely close to (II) and F_2' is the shear in span (II)-(III) infinitely close to (II).

Here equation (10) gives $F_2' = \frac{M_3 - M_2}{l'} + \frac{w'l'}{2}$.

The similar equation for span (I)-(II),

$$M_1 = -\frac{wl^2}{2} + F_2 l + M_2,$$

gives
$$F_2 = \frac{M_1 - M_2}{l} + \frac{wl}{2},$$

and
$$R_2 = F_2' + F_2 = \frac{M_3 - M_2}{l'} + \frac{M_1 - M_2}{l} + \frac{wl + w'l'}{2}.$$

Another method of finding the reactions is illustrated in the following solution of Ex. 106, page 97.

Referring to Fig. 51, and applying the theorem to the spans 1-2 and 2-3, we have

$$M_1 l + 2\,M_2\,(2\,l) + M_3 l = -\frac{wl^3 + wl^3}{4}. \quad \quad (1)$$

Similarly for spans 2-3 and 3-4,

$$M_2 l + 2 M_3 (2 l) + M_4 l = -\frac{wl^3}{2}. \quad . \quad . \quad (2)$$

Thus two equations and four unknowns.

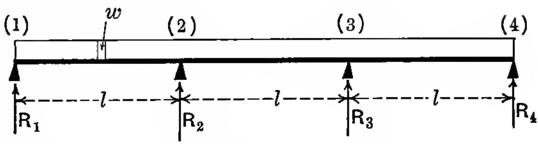


FIG. 51

But in this case the B.M. at 1 is zero, this being a free end,

or $$M_1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

also $$M_4 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Solving equations (1), (2), (3), and (4), we have from

(1) $$4 M_2 + M_3 = -\frac{wl^2}{2},$$

from (2) $$M_2 + 4 M_3 = -\frac{wl^2}{2},$$

so that $$M_2 = -\frac{wl^2}{10} \quad \text{and} \quad M_3 = -\frac{wl^2}{10}.$$

From Fig. 51, by the definition of bending moments

$$M_2 = R_1 l - \frac{wl^2}{2}, \text{ but it also equals } -\frac{wl^2}{10},$$

so that $$R_1 = +\frac{2 wl}{5};$$

and taking moments about (3), Fig. 51,

$$R_1 (2 l) + R_2 (l) - \frac{w (2 l)^2}{2} = M_3 = -\frac{wl^2}{10},$$

whence $$R_2 l = -\frac{wl^2}{10} + 2 wl^2 - \frac{4 wl^2}{5} = \frac{11 wl^2}{10},$$

and $$R_2 = \frac{11 wl}{10}.$$

EXERCISE 107. Find the reactions for a beam continuous over four equal spans, length of each span l inches, and bearing a uniformly distributed load of w pounds per inch run.

EXERCISE 108. A continuous beam (spans l_1, l, l_1) is loaded uniformly on the middle span only. Find the reactions at the end abutments.

EXERCISE 109. Sketch the diagrams of S.F. and B.M. for the beam described in Ex. 108.

EXERCISE 110. Sketch the diagrams of S.F. and B.M. for the beam described in Ex. 106.

EXERCISE 111. In Ex. 106, assume the load over each span supported by a simple beam and draw the S.F. and B.M. diagrams under this condition.

The full lines in Fig. 52 show at (a) the shearing force and at (b) the bending moment diagrams for the continuous beam described in Ex. 107.

The sucessive reactions at the supports are

$$\frac{11}{28}wl,\ \frac{32}{28}wl,\ \frac{26}{28}wl,\ \frac{32}{28}wl,\ \frac{11}{28}wl.$$

The shearing forces at the beginning and end of each span are

$$\frac{11}{28}wl,\ -\frac{17}{28}wl;\ \frac{15}{28}wl,\ -\frac{13}{28}wl;$$

$$\frac{13}{28}wl,\ -\frac{15}{28}wl;\ \frac{17}{28}wl,\ -\frac{11}{28}wl.$$

The bending moments at the beginning of each span, the maximum bending moment in each span, and the bending moment at the end of each span are

$$0,\ \frac{121}{1568}wl^2,\ -\frac{168}{1568}wl^2;\ -\frac{168}{1568}wl^2,$$

$$\frac{57}{1568}wl^2,\ -\frac{112}{1568}wl^2;\ -\frac{112}{1568}wl^2, \text{etc.}$$

If the same load were carried by simple beams (i.e., if the continuous beam were cut over each support) the shearing force and the bending moment diagrams would

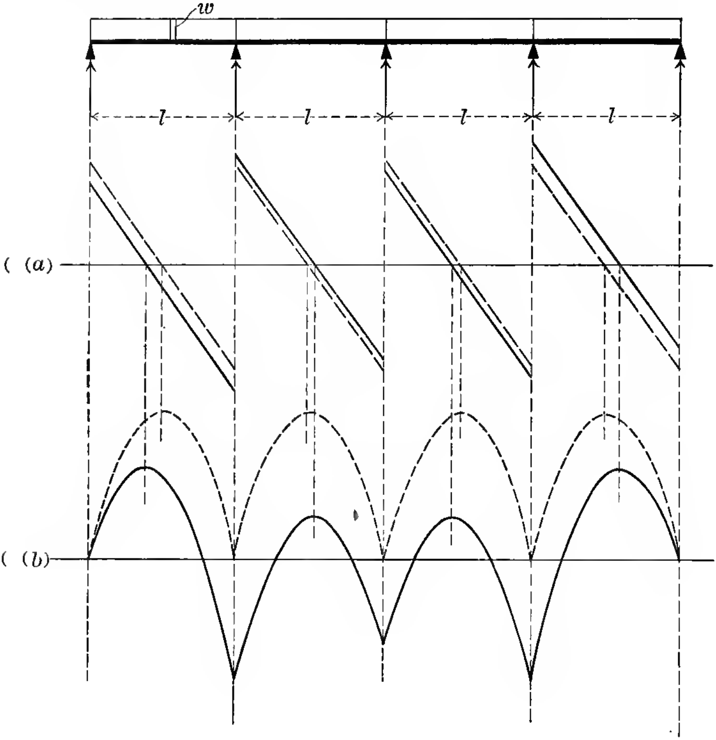


FIG. 52

change to those shown by the dotted lines in Fig. 52 (a) and (b).

Here the greatest shear is $\frac{14}{28}wl$ and the maximum bending moment is $\frac{196}{1568}wl^2$ at mid-span in each span.

These diagrams illustrate the following facts: (a) that the greatest bending moments in the continuous beam occur over the supports and in simple beams at mid-span, (b) that the greatest and the average bending moments are less for the continuous beam than for the simple beams, (c) that the shearing force changes but slightly when the continuous beam is replaced by simple beams.

As continuous beams are usually built-up girders of varying cross section, it follows that the heaviest portions of a continuous beam lie over or near the supports, and thus the weight of the girder does not materially increase the bending moment, as it must do in the case of the simple beams, which must be made strongest and thus heaviest at mid-span. Also, the weight of the continuous beam would on the whole be less than the sum of the weights of the separate simple beams.

To counterbalance these advantages, the use of the continuous beam has serious disadvantages. It should be noted that the slightest change in level in the abutments or want of straightness in the beam changes the conditions governing the computation of the bending moments. Also, moving loads passing over the beam will cause the points of inflection (zero bending moment) to shift their positions, and thus near such points the bending moments must change from plus to minus, or vice versa. The greater the moving load as compared to the permanent load on the structure, the greater this disadvantage; so that the longer the spans, the less will this effect appear, for under these conditions the weight of the structure forms the greater part of the total load.

Owing to the above disadvantages, continuous beams are now seldom used.

In cantilever bridges, pin connections are made at some of the points at which the points of inflection would occur if the bridge were replaced by a continuous, uniformly loaded beam. These pin connections transmit only shearing forces, so that the bending moments at such points must remain zero, no matter how the bridge may be loaded. This construction simplifies the calculation of the stresses (the structures being now statically determinate) and at the same time does away with the disadvantages inherent to continuous structures.

At which points in the continuous beam described in Ex. 106 could hinges or pin connections be introduced, as indicated in the above paragraph? Could hinges be placed at all points of inflection in Ex. 106?

CHAPTER V

STRUTS AND COLUMNS

SECTION XI

ECCENTRIC LONGITUDINAL LOADS (SHORT COLUMNS)

SHORT columns have already been discussed under axial loads (page 11). Here the resultant pressure passed through the center of area of the section; the unit stress produced was found by dividing the total force by the area of the section upon which it acts.

A short column loaded parallel to its axis, but also eccentrically, the resultant load no longer passing through the center of area of the section but still passing through an axis of symmetry of the section, will now be considered.

In Fig. 53, showing a plan and elevation of the column considered, P is the eccentric load, and e **its eccentricity.** Consider two equal and opposite forces, P, (dotted in Fig. 53) applied at the center of area O of the section; these will not disturb the conditions of the problem. We now have a **central load** P and a **couple of moment** Pe acting upon the section. The central load, P, produces a uniform compression, $p_c = \frac{P}{A}$, throughout the section, where A is the area of the section. This stress is represented in Fig. 54 by the dotted rectangle.

The couple (moment Pe) acts upon the section in exactly the same manner as the bending moment acts upon a beam section, so that the formula $M = \frac{pI}{y}$ applies and the stress calculated by its means must be added to the direct stress $p_c = \frac{P}{A}$.

For the present calculation I may be conveniently expressed in terms of the "radius of gyration," k, of the section by means of the **defining equation I = Ak²**. The neutral axis, NA, about which I and k are to be reckoned, must evidently pass through O and be perpendicular to GH, the axis of symmetry of the section. The stress due to the eccentricity of the load is thus

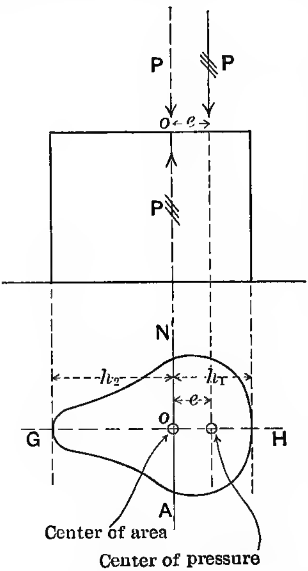


FIG. 53

$$p = \frac{My}{I} = \frac{(Pe)y}{Ak^2} = p_c \frac{ey}{k^2},$$

where y is the distance from NA, the neutral axis, to that element of the section on which the stress is p. When y is measured to the right of NA, in Fig. 53, p is evidently a compressive stress, and when toward the left a tensile stress. Moreover, the variations of this stress follow the straight-line law and are plotted in Fig. 54 as a dotted-dashed line. The total combined stress, shown by the full line, is obtained by adding the stress, p_c, due to direct compression, to the stress, p, due to the eccentricity of the load.

The greatest stress on the section (at the extreme right) is thus a compression

$$p_1 = p_c + p_{y=h_1} = p_c + p_c \frac{eh_1}{k^2} = p_c\left\{1 + \frac{eh_1}{k^2}\right\} = \frac{P}{A}\left\{1 + \frac{eh_1}{k^2}\right\},$$

and the least stress on the section (at the extreme left) is

$$p_2 = p_c + p_{y=-h_2} = p_c - p_c \frac{eh_2}{k^2} = p_c\left\{1 - \frac{eh_2}{k^2}\right\} = \frac{P}{A}\left\{1 - \frac{eh_2}{k^2}\right\};$$

if this expression becomes negative, then the stress becomes tensile.

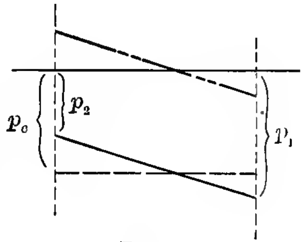


FIG. 54

EXERCISE 112. (a) What is the stress in the above case at the center of area of the section? (b) Along the neutral axis of the section? (c) At any point of the section?

EXERCISE 113. (a) At what distance from *NA*, Fig 53, is the stress zero? (b) Under what condition will the stress in some parts of the material become tensile?

The **line parallel to NA**, the neutral axis of the section, along which **no stress** occurs, is called **the neutral axis of stress.**

EXERCISE 114. In what ratio is the strength of the column reduced by the eccentricity of the load?

EXERCISE 115. If the section is rectangular and $h \times b$, what is the greatest eccentricity allowable if no tension is to result?

The above exercise shows that to **avoid tension** on a rectangular section of a **material in compression** the resultant load must act **within the middle third.** (Prove this.)

Exercise 116. What is the corresponding rule for a hollow circular section having as internal and external radii r and R respectively?

Exercise 117. On a short cast-iron column, 6 inches external and 5 inches internal diameter, the load is 20,000 pounds and the center of pressure lies a half-inch from the center of the section. Find the greatest and least unit stress.

Exercise 118. The vertical column of a crane is an I-beam 25 inches deep, section area 24 square inches, I about a neutral axis parallel to the flanges is 3000 inches^4. Find the greatest and the least unit stress due to a load of 20,000 pounds carried 14 feet from the center of the section of the column.

The above theory can be applied to "cranked" bars, machine frames, short columns, etc.; but in the case of columns in which the length is more than 8 to 10 times the least width the theory given on page 111 must be applied. In the case of crane hooks or any bars of considerable curvature the method outlined above must not be used; this theory giving results nearly 50 per cent from the truth in some cases.

As an application to the stability of masonry structures in which tension must always be avoided, solve the following exercise:

Exercise 119. The section of a masonry dam, with a vertical face subject to water pressure, has a height h and a thickness, t_2 at top and t_1 at bottom. Assuming the water level at the top of the dam, the vertical section of the dam trapezoidal, and the specific gravity of the masonry as s, find the height of the dam so that the resultant pressure acts at the outer middle third of the base of the section.

Section XII

BUCKLING (LONG COLUMNS)

A column or strut is a bar under compressive loads. If such columns are composed of homogeneous material, if they are originally perfectly straight, and if the loads are axially applied, then theoretically the column would fail by crushing. Practically these conditions are never realized, and columns whose length exceeds but a few times their least lateral dimension show a tendency to buckle long before the crushing stress is reached. This buckling always takes place in that lateral direction which is perpendicular to the neutral axis of the section about which the moment of inertia is least, this being the weakest direction of the column. It is thus advisable, as a saving of material, to have the moments of inertia about the two principal axes of the section of the column equal to each other.

Exercise 120. Two joists, each 2 inches by 4 inches, are to be placed 6 inches apart between centers, and connected by two other joists, each 8 inches by x inches, so as to form a hollow rectangular column. Find the proper value of x.

Exercise 121. Two I-beams, each having principal moments of inertia of 14.62 inches^4 and 441.7 inches^4 and a sectional area of 12.48 inches^2, are to be used as a column. How far apart center to center should they be placed?

The effect of the lacing holding these I-beams together is to be neglected in the calculation.

The theory of column strength is unfortunately very incomplete. This is due principally to the numerous practical conditions involved, such as the slight variation from perfect straightness in the unloaded column, the end conditions, etc., which cannot be subjected to mathematical investigation.

The discussion in this section will be divided into, first, a discussion of formulas having a theoretical basis, and then, a consideration of some empirical formulas.

Euler's Formula. — Consider first the oldest, and only theoretical formula, deduced by Euler in 1757. This discussion deals with long, slender columns perfectly straight, of homogeneous material, and with the load axially applied so that owing to the load no buckling could ever occur. Such columns are called **ideal columns.**

Euler assumed that an ideal column would, when lightly loaded and then deflected (buckled) by a lateral force, return to its original condition of perfect straightness when the deflecting force is removed. That particular load under which the column would fail to straighten after a removal of the deflecting force he called the **critical load.** An ever so slight increase of the load beyond its critical value would cause an ever-increasing deflection, and finally failure by buckling. This critical load is the value we wish to find.

In any discussion of column formulas the **end conditions** are of great importance. Fig. 55 illustrates these conditions and the corresponding manner of buckling is indicated by dotted lines. (I) shows a column with round or pin-connected ends, (II) a column with one end fixed and the other end pin-connected, (III) both ends fixed, (IV) one end free, the other fixed. These cases will be referred to as Cases I, II, III, and IV, respectively.

Consider Case I, the pin-ended column; for convenience assume the column in a horizontal position (when

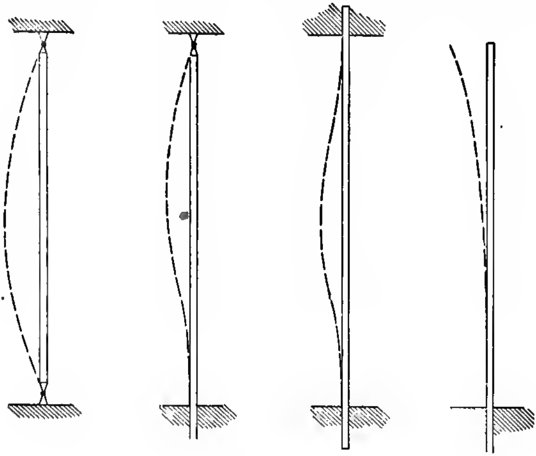


FIG. 55

it may more properly be called a strut) and weightless (Fig. 56). The load P is to be the critical load, so that the column displaced laterally remains in equilibrium

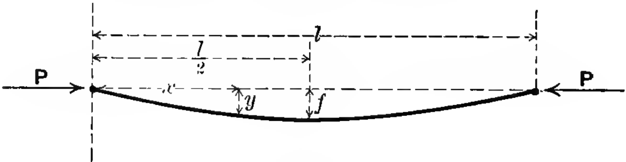


FIG. 56

under any displacement, neither increasing in deflection nor straightening itself as it would under a load less than P.

Assuming that $EI\left(\frac{d^2y}{dx^2}\right) = M_x$, the bending moment at any point, by analogy to the case of a bent beam, we have

$$EI\frac{d^2y}{dx^2} = P(-y) = -Py,$$

or

$$2\,EI\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} = -2\,Py\frac{dy}{dx}.$$

Integrating, we have

$$EI\left(\frac{dy}{dx}\right)^2 = P(-y^2 + C_1),$$

or

$$\frac{dy}{dx} = \sqrt{\frac{P}{EI}}\sqrt{C_1 - y^2}.$$

Separating the variables

$$\sqrt{\frac{P}{EI}}\,dx = \frac{dy}{\sqrt{C_1 - y^2}},$$

and integrating again

$$\sqrt{\frac{P}{EI}}\,x = \sin^{-1}\frac{y}{\sqrt{C_1}} + C_2.$$

Now let the maximum deflection due to buckling be f, then $\frac{dy}{dx} = 0$ when $y = -f$, in which case $C_1 = f^2$, and $y = 0$ when $x = 0$, whence $C_2 = -\sin^{-1} 0 = -n\pi$, where n is any integer. So that

$$\sqrt{\frac{P}{EI}}\,x = \sin^{-1}\frac{y}{f} - n\pi,$$

or

$$y = f\sin\left\{\sqrt{\frac{P}{EI}}\,x + n\pi\right\} \quad . \quad . \quad . \quad . \quad (1)$$

As yet P has not been found and our equation contains a parameter f which may take any value without affecting P, the critical load.

Now as $y = 0$ when $x = l$,

we have $$\sqrt{\frac{P}{EI}}\,l = \sin^{-1} 0 - n\pi,$$
$$= m\pi - n\pi,$$
$$= (m - n)\pi,$$
where m is any integer.

Thus $$P = \frac{\pi^2 (m - n)^2 EI}{l^2} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and by substitution in equation (1)

$$y = f \sin \left\{ \frac{\pi (m - n)}{l} x + n\pi \right\} \quad . \quad . \quad . \quad (3)$$

If equation (3) be plotted the various forms which a buckling strut may assume will be found. To do this, let us assume $n = 0$, and m equal successively to 0, 1, 2, 3.

The first set of values $n = 0$, $m = 0$ give $P = 0$, $y = 0$. This of course means that no buckling occurs under zero load.

Next $n = 0$ and $m = 1$ give

$$P = \frac{\pi^2 EI}{l^2} \quad \text{and} \quad y = f \sin \left(\frac{x}{l} \pi \right), \quad . \quad . \quad . \quad \text{(a)}$$

then $n = 0$ and $m = 2$ give

$$P = \frac{4\pi^2 EI}{l^2} \quad \text{and} \quad y = f \sin \left(\frac{2\pi x}{l} \right), \quad . \quad . \quad \text{(b)}$$

and $n = 0$ and $m = 3$ give

$$P = \frac{9\pi^2 EI}{l^2} \quad \text{and} \quad y = f \sin \left(\frac{3\pi x}{l} \right), \quad . \quad . \quad \text{(c)}$$

The curves corresponding to equations (a), (b), and (c) are plotted in Fig. 57.

Exercise 122. Investigate equations (2) and (3) above for $m = 1$, $n = 0, 1, 2, 3$.

From the above results the following inferences can be drawn. That for

Case I. (Pin-connected ends.)

$P_1 = \frac{\pi^2 EI}{l_1^2}$ is the critical load (see page 118) deduced from equations and figure (a), where l_1 is the length of the column in inches.

Case II. (one end fixed, the other pin-connected) can be approximated from Fig. 57 (b) and the corresponding

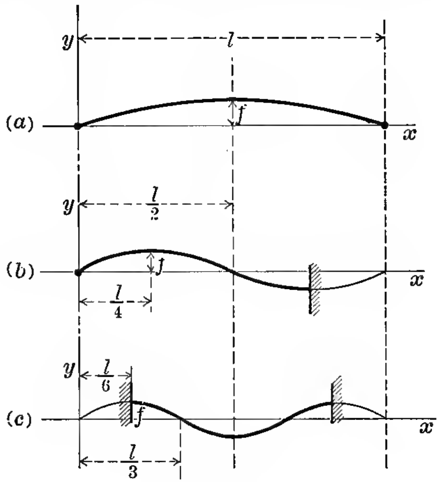


FIG. 57

equations when $n = 0$ and $m = 2$ if $l_2 = \frac{3}{4} l$, l_2 being now the length of the column or strut in inches.

Thus $$P_2 = \frac{9\,\pi^2 EI}{4\,l_2^2}$$ (see page 118).

It should be noted that in this approximation the pin end is not in line with the fixed end, the lateral displacement being f.

A more exact calculation* (beyond the scope of this book) gives $\frac{2.046\,\pi^2 EI}{l^2}$ as the critical load.

Case III. (Both ends fixed.) From Fig. 57 (c) and the corresponding equations, placing $l_3 = \frac{2}{3}\,l$, where l_3 is now the length of the strut in inches, we have

$$P_3 = \frac{4\,\pi^2 EI}{l_3^2}$$ as the critical load (see page 118).

Case IV. (One end fixed, the other free.) From Fig. 57 (a) and the corresponding equations when $l_4 = \frac{l}{2}$ we have

$$P_4 = \frac{\pi^2 EI}{4\,l_4^2}$$ (see page 118),

l being the length of the strut in inches.

Exercise 123. Obtain the critical load for Case II from Fig. 57 (c).

Exercise 124. Obtain the critical load for Case I from (1) Fig. 57 (b); (2) Fig. 57 (c).

Exercise 125. A solid pin-connected steel column 6 inches in diameter, 37 feet long, can bear what critical load?

Exercise 126. A square wooden column with flat ends is to be 20 feet long and carry a load of 9500 pounds. Compute its size for a "factor of safety" of 10.

Slenderness Ratio of a Column. — Euler's Equations may be changed to the following form:

$$P = c\,\frac{EI}{l^2} = c\,\frac{EAk^2}{l^2} = \frac{cEA}{\left(\frac{l}{k}\right)^2},$$

where k is the minimum "radius of gyration."

* Grashof, Theorie der Elasticität und Festigkeit.

The ratio $\left(\frac{l}{k}\right)$ is known as the **slenderness ratio** of the column.

Euler's Equations should never be applied unless the slenderness ratio of the column is greater than 150. This limit has been established by numerous experiments which show that the values of P obtained from Euler's Equations for shorter columns are much too large.

Exercise 127. Compute the slenderness ratio for Exs. 125 and 126.

Rankine's Formula. — This formula, obtained in various ways by Navier and Schwarz as well as Rankine, is the result of an attempt to produce a formula which will give satisfactory results for columns having a smaller slenderness ratio than the columns to which Euler's Equations apply.

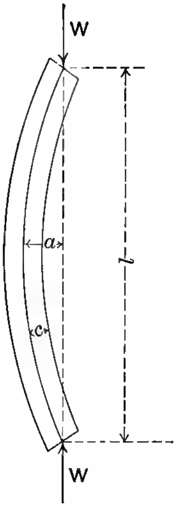


Fig. 58

Another objection to Euler's Equations is the fact that it introduces no constant for the strength of the material.

To develop Rankine's Formula, let

W = the load upon the column,
A = its sectional area,
I = its minimum principal moment of inertia,
k = the corresponding radius of gyration,
c = the distance of the extreme compression fiber from the corresponding neutral axis,
a = the maximum deflection due to buckling,
p = greatest compressive stress in the column.

Then a may be regarded as the **eccentricity** of the load (Fig. 58), and the compressive stress due to this eccentricity is $p' = \frac{Wac}{I}$; adding to this the direct compressive stress $p'' = \frac{W}{A}$, the greatest compressive stress in the column is

$$p = \frac{Wac}{I} + \frac{W}{A} = \frac{W}{A}\left(1 + \frac{ac}{k^2}\right),$$

whence
$$W = \frac{Ap}{1 + \frac{ac}{k^2}}.$$

If we now assume the same law for buckling as for bending, then the maximum deflection for the column is

$$a = \frac{c_1 p''' l^2}{Ec} \quad \text{(see page 78)},$$

where p''' is the stress due to bending or buckling and c_1 is a constant depending upon the end conditions.

Thus

$$W = \frac{Ap}{1 + \frac{c_1 p''' l^2}{Ek^2}} = \frac{Ap}{1 + C\left(\frac{l}{k}\right)^2}.$$

This is Rankine's Formula. Here C is a constant which must be determined by experiments upon columns of various materials and variously fixed at the ends, and p is the compressive working stress for the material considered.

The **values of C for Rankine's Formula** are for timber $\frac{1}{3000}$, cast iron $\frac{1}{6000}$, wrought iron $\frac{1}{36000}$, and for steel $\frac{1}{30000}$ if the ends are both fixed. **If one end is pin-connected and the other fixed the above constants must be multiplied by** 1.78, **and for columns both ends pin-connected the multiplier is** 4.

Rankine's Formula may be used with the above constants for values of $\left(\frac{l}{k}\right)$ between 20 and 200.

EXERCISE 128. A hollow wooden column of rectangular section, outside dimensions 4 inches by 5 inches, inside 3 inches by 4 inches, length 18 feet, is to carry 5400 pounds; the ends are fixed. Find the greatest fiber stress.

EXERCISE 129. Find the safe load for a fixed-ended timber column 3 inches by 4 inches, 10 feet long, if the allowable stress is 800 pounds per square inch. If the column were "very short" what would be the safe load?

EXERCISE 130. Find the size of a square wooden column, fixed at the ends, and 24 feet long, to carry 100,000 pounds, the factor of safety to be 10.

EXERCISE 131. The diameter of a piston is 18 inches and the greatest steam pressure 150 pounds per square inch. Find the diameter of a steel piston rod 5 feet long, factor of safety 10. This compression member is to be considered pin-connected. Why? What is the slenderness ratio of this rod?

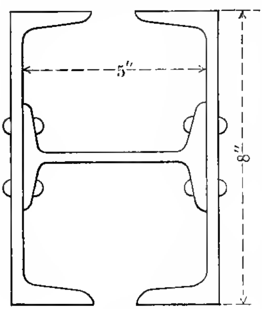


FIG. 59

EXERCISE 132. A column is built up of two channels and an I-beam as shown in Fig. 59. The properties of the channels and I-beam are shown in Fig. 60; these are taken from the handbook of the Carnegie Steel Company. (a) Compute the two principal "moments of inertia" of this section. (b) If the unsupported length of this column is 15 feet and its ends are fixed, what load can it safely carry?

Rankine's Formula for the strength of columns may be called a semirational one. Many entirely empirical formulas have been devised. Of these may be mentioned J. B. Johnson's parabolic formula* and Thomas H. Johnson's straight-line formula†.

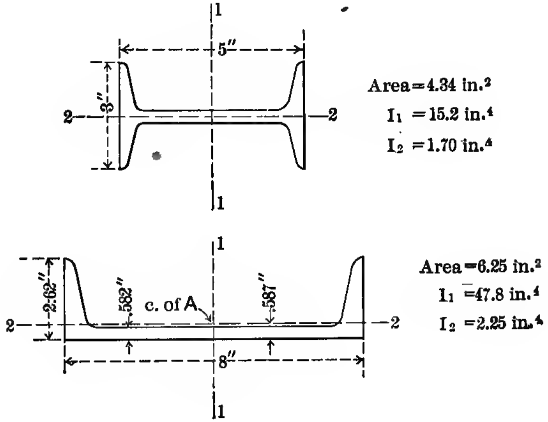


FIG. 60

The straight-line formula, a very convenient one for general use, is usually stated as follows:

For steel columns in buildings, factor of safety included,

$$p = 16{,}000 - 60\left(\frac{l}{k}\right), \text{ for both ends pin-connected,}$$

$$p = 16{,}000 - 57\left(\frac{l}{k}\right), \text{ for one end pin-connected, the other fixed,}$$

$$p = 16{,}000 - 45\left(\frac{l}{k}\right), \text{ for both ends fixed.}$$

* Johnson's Framed Structures, 8th ed., 1905, pp. 159–171, and Trans. Amer. Soc. Civil Engrs., Vol. XV., pp. 518–536.

† Trans. Amer. Soc. Civil Engrs., 1886, p. 530.

For steel columns in bridges, factor of safety included,

$$p = 16{,}000 - 80\left(\frac{l}{k}\right), \text{ both ends pin-connected,}$$

$$p = 16{,}000 - 60\left(\frac{l}{k}\right), \text{ both ends fixed.}$$

In these equations p is the safe working stress in pounds per square inch of column section, l is the unsupported length in inches, and k is the least radius of gyration of the column section in inches; moreover, $\left(\frac{l}{k}\right)$ should not exceed 100.

The following are forms of Rankine's Formula much used for obtaining the safe dead load, W, in pounds in terms of the sectional area of columns in square inches:

$$W = \frac{16{,}000\,A}{1 + \dfrac{l^2}{14{,}000\,k^2}}, \text{ for soft steel,}$$

and

$$W = \frac{17{,}000\,A}{1 + \dfrac{l^2}{11{,}000\,k^2}}, \text{ for medium steel.}$$

Here no allowance is made for various end connections.

CHAPTER VI

TORSION

Section XIII

STRESS AND STRAIN DUE TO TORSION

Straight bars have been considered under the action of forces producing compression, tension, flexure, and buckling. In this chapter the effect of equal and opposite couples applied to the ends of a bar, to whose axis the axes of the couples are to be parallel, is to be considered. Such couples **twist** the bar and the bar is subjected to **pure torsion.**

The following discussions apply only to **bars of circular section,** the consideration of bars of other sections being beyond the scope of this book.*

In Fig. 61 let AB represent an element of the cylindrical surface of the bar before twisting occurs, then under the action of the couples shown each section of the cylinder from P to O is displaced angularly about the axis OP relatively to the previous one by a differential angle. These displacements accumulate and amount at O to the angle BOC called the **angle of twist.** The element AB is at the same time distorted into the **helix** AC, which is inclined to the axis OP at a constant angle ϕ, called the **angle of torsion.**

* For such discussion see Elastizität and Festigkeit, C. Bach, Berlin, 1905, pp. 302–357.

The stress existing between two adjacent sections of the bar is evidently one of shear and the angle ϕ measures the deformation or strain resulting from this shearing

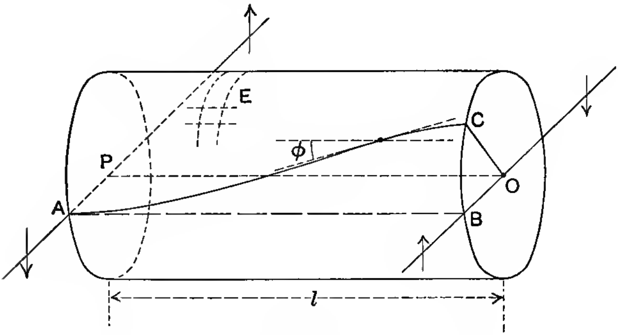


FIG. 61

stress, so that ϕ is also known as the **angle of shear** (see page 126). At each section perpendicular to the axis of the bar a **resisting torque** must act. This resisting torque balances the applied torque due to the external couple.

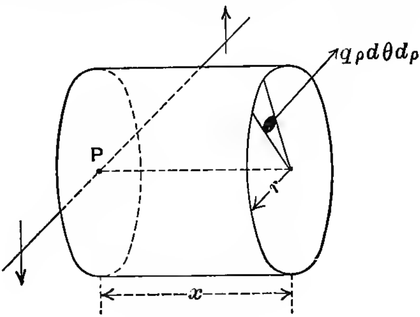


FIG. 62

To find this resisting torque, consider a portion of the bar, x inches long and including the end P, as a free body (Fig. 62). The material to the right of the section at x (removed to form the free body) will exert a force on each and every element, $\rho d\rho d\theta$, of the section equal to $q\rho d\rho d\theta$, where q is the unit shearing stress. This force will be perpendicular to the radius, ρ, of the element considered and must lie in the plane of the section. Also the sum of the moments of these forces constitutes the resisting torque at the section and must balance the torque due to the couple at P. Thus if T represents the torque of the couple applied at P or the equal moment of the couple applied at O, we have

$$T = \int\int (q\rho d\rho d\theta)\, \rho, \quad . \quad . \quad . \quad . \quad (1)$$

q is evidently a variable quantity, being zero at the axis and increasing towards the surface, where it reaches its greatest value. As the strain is assumed proportional to the distance from the axis of the cylinder, the stress may be considered to follow the same straight-line law, and we may place

$$q = \frac{q_r \rho}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where q_r is the shearing stress at the surface of the cylinder, where the radius is r.

Substituting from (2) into (1), we have

$$T = \frac{q_r}{r}\int\int \rho^2\,(\rho d\theta d\rho).$$

The expression $\int\int \rho^2\,(\rho d\theta d\rho)$ is by definition the polar "moment of inertia" of the section, or, better, the polar second moment of area. If this is designated by I_p, we have

$$\mathbf{T} = \frac{\mathbf{q}_r\mathbf{I}_p}{\mathbf{r}} = \frac{\mathbf{q}\mathbf{I}_p}{\rho}.$$

The analogy of this formula to $M = \dfrac{pI}{y}$ should be noted.

EXERCISE 133. Show that the polar moment of inertia of a circular section is $\frac{\pi r^4}{2}$.

EXERCISE 134. Compare the strength of a circular hollow shaft with that of a solid one having the same sectional area. Let d_1 and d_2 be the outside and inside diameters of the hollow shaft.

EXERCISE 135. What is the ratio of the strength of the hollow shaft to that of the solid one when $d_1 = 2\,d_2$?

In order to find the angle of twist or the total strain due to the torsion, the **modulus of elasticity of shear** must be defined. Let the square in Fig. 63 represent any element of the cylindrical surface of the bar before distortion, as represented by the dotted lines at E in Fig. 61. Then under the action of the shearing stresses, q, the displacement represented by the dotted lines in Fig. 63 occurs, and the angle of shear, ϕ, measures this displacement or strain; for the total strain between the two surfaces at a distance n from each other is m, so that the strain between two surfaces at unit distance from each other would be $\frac{m}{n}$, or the unit strain is $\frac{m}{n} = \tan\,\phi$, and as ϕ is very small we may place $\tan\,\phi = \phi$.

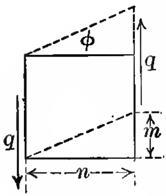


FIG. 63

ϕ is thus the unit strain, and analogously to $\frac{p}{s} = E$ we put

$$\frac{q}{\phi} = G,$$

where G is the modulus of elasticity of shear.

To obtain the **angle of twist**, consider a portion of the bar dx inches long (Fig. 64). Here AB represents the helix and ϕ the angle of shear. The angle DHC ($= \beta$) is the twist in a length x and $d\beta$ the increment in the twist per increment, dx, of the length.

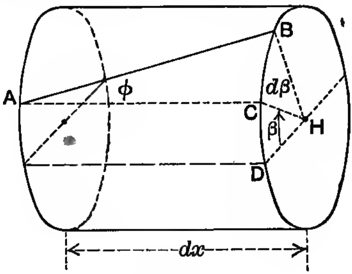


Fig. 64

Now the arc $BC = \phi dx = r d\beta$,

or $$d\beta = \frac{\phi dx}{r}, \text{ but } \phi = \frac{q_r}{G} \text{ and } q_r = \frac{Tr}{I_p},$$

so that $$d\beta = \frac{T dx}{I_p G},$$

and the total angle of twist, θ, in a bar l inches long is

$$\theta = \frac{T}{I_p G}\int_0^l dx = \frac{Tl}{I_p G}.$$

Exercise 136. A steel shaft $2\frac{1}{2}$ inches in diameter carries a pulley 30 inches in diameter; the difference in tensions on the two parts of the belt is 2000 pounds. Find the greatest unit shear and the angle of twist at a distance of 30 feet from the pulley.

Exercise 137. Compare the stiffness of a hollow and a solid shaft having the same sectional area and the same length (see Ex. 134).

Exercise 138. What is the ratio of the stiffness of the hollow shaft to that of the solid one when $d_1 = 2\, d_2$?

SECTION XIV

APPLICATIONS

Shaft Couplings. — Shafts are sometimes joined by fastening two flanges, forming parts of the shafts to be connected, by means of bolts, as shown in Fig. 65. Bolts so used will be subject to shear whenever a torque is transmitted from one shaft to the other.

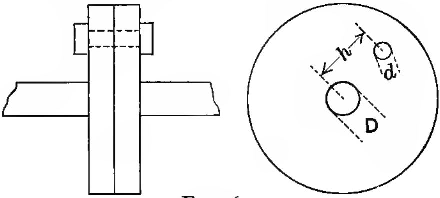


FIG. 65

In order to find the number of bolts, if their diameter is known, or their diameter, if the number of bolts is given, which will make the couplings equal in strength to the shafts,

let d be the diameter of the bolts,

D be the diameter of the shaft,

h be the distance between the center of shaft and the centers of the bolts,

and n the number of bolts.

Then the resisting moment of the shaft,

$$\frac{q_r I_p}{r} = \frac{q_r \pi D^3}{16},$$

should equal the resisting moment of the bolts,

$$n \frac{q_r \left\{ \frac{\pi d^4}{32} + \frac{\pi d^2 h^2}{4} \right\}}{\frac{d}{2} + h} \quad \text{(see Ex. 139)},$$

thus, we have $D^3(d + 2h) = nd^2(d^2 + 8h^2)$. As an approximation, when d is small as compared to h, $D^3 = 4nd^2h$.

Show that this approximate solution may also be obtained by considering the shearing stress in the bolts of constant magnitude and uniformly distributed over their sections.

EXERCISE 139. Show that the polar "moment of inertia" of a section about an axis parallel to the polar axis through its center of area equals its polar "moment of inertia" about the polar axis through its center of area plus the product of the area of the section by the square of the distance between the axes.

EXERCISE 140. A hollow shaft, 17 inches outside and 11 inches inside diameter, is to be coupled by 12 bolts placed on a circle 40 inches in diameter. What should be the diameter of the bolts?

Power Transmitted by Shafts. — Problems involving the power transmitted by shafts of circular section are easily solved if the following facts are remembered.

(a) Work in foot-pounds = (torque in pound-feet) × (angular displacement in radians).

(b)
$$T = \frac{q_r I_p}{r},$$

where T is measured in **inch-pounds.**

EXERCISE 141. Show that if H is the H.P. to be transmitted at n revolutions per minute, and q_r is the working strength of the material of the shaft for shearing stress, then the diameter of the shaft in inches is

$$d = \sqrt[3]{\frac{3,170,000\,H}{\pi^2 n q_r}}.$$

EXERCISE 142. What H.P. can a cast-iron shaft 3 inches in diameter transmit at 15 r.p.m. with a factor of safety of 15?

EXERCISE 143. What should be the diameter of a structural steel shaft to safely transmit 500 H.P. at 200 r.p.m.?

EXERCISE 144. Find the factor of safety for a wrought-iron shaft 3 inches in diameter when transmitting 40 H.P. at 100 r.p.m.

EXERCISE 145. A steel wire 0.18 inch in diameter and 20 inches long is twisted through an angle of 18.5 degrees by a torque of 20 inch-pounds. Determine its shearing modulus of elasticity.

EXERCISE 146. A steel shaft transmits 50 H.P. at 200 r.p.m. If its length is 20 feet and diameter 3 inches, through what angle is this shaft twisted?

EXERCISE 147. 90 H.P. at 120 r.p.m. are to be transmitted through a wrought-iron shaft. What must be the diameter so that the angle of twist shall not exceed 1 degree in a length of 8.5 feet?

What is the factor of safety under these conditions?

CHAPTER VII

STRESS, STRAIN, AND ELASTIC FAILURE

SECTION XV

STRESS

THE object of this chapter is a general discussion of stress and strain, and, in particular, a discussion of the effect produced by the simultaneous occurrence of several stresses.

Stress. — In all previous work only the stresses across certain planes, more or less arbitrarily selected, were considered. Thus, across a plane section perpendicular to the axis of a bar in tension the stress was found to be **a normal stress,** and across a plane section perpendicular to the axis of a bar in torsion the stress was found to be a **tangential** or **shearing stress.** In each of these cases a single stress acted alone; in the first we have pure normal stress, in the other pure shearing stress. It is, however, apparent that very different results would have been obtained had the sections been passed in any direction other than perpendicular to the axis of the bars. Under such conditions it is very likely that both normal and shearing stresses would have been found to act across the section, or, to state the case different y, the actual stress across the section would have been found to be oblique.

In general it may readily be conceived that the stress across any section taken at random within a stressed material would be oblique to the section as shown in Fig. 66, where the arrow S represents the stress across the plane MN.

An oblique stress, S, can always be resolved into two component stresses, p and q, the first normal, and the second tangential to the plane MN. It is often convenient again to resolve q into the components q' and

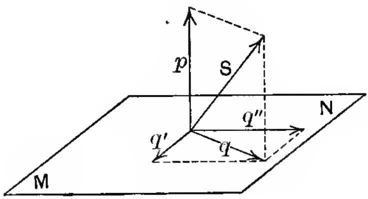


FIG. 66

q'' (both in the plane MN) in such a manner that p, q', and q'' shall each be parallel to one of the rectangular axes to which the location of the plane MN is referred.

Conceive now a differential element, $dx\,dy\,dz$, in a stressed body as represented in Fig. 67. Let the oblique stresses across its various faces be resolved into components parallel to the axes of reference, then the six possible oblique stresses must be represented by the eighteen normal and shearing stresses shown in Fig. 67.

The notation used in designating these numerous stresses is as follows:

Note first that a double subscript is employed. **The first of these subscripts** always agrees with the name of the axis to which the **plane** across which the stress occurs is **perpendicular**; the **second subscript** always denotes the axis to which the stress is parallel. Thus the stresses across face 3 have as first subscript x, for their plane $dy\,dz$ is perpendicular to the **X**-axis (thus, q_{x-}, q_{x-}, p_{x-}); the second subscript indicating the axes to which the stresses are parallel is now added (thus, q_{xy}, q_{xz}, p_{xx}). In the case of the normal stress p_{xx}, as

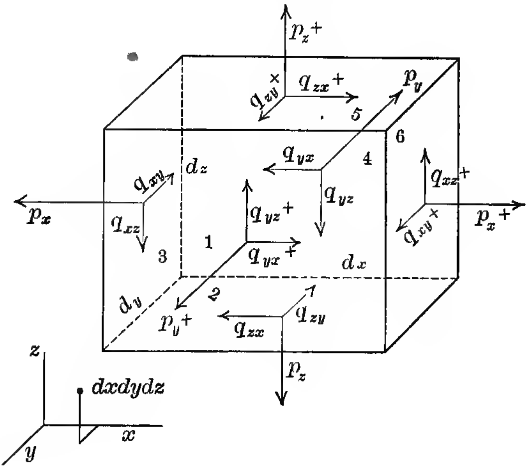


FIG. 67

there is no need of the second subscript, there being no other p_{x-}, it is usually omitted, so that p_{xx} is written p_x.

The reader should now write the names of the stresses on faces 1 and 2 and compare his results with those in Fig. 67.

Consider the stresses on the face 6. The normal stress on this face is denoted by p_x. But as it acts at a differential distance dx from face 3, this stress will not in general be equal to the stress p_x on face 3, for it may have changed by the partial differential of p_x with respect to x, $\left(\frac{\partial p_x}{\partial x}dx\right)$, and it should be denoted by $p_x + \frac{\partial p_x}{\partial x}dx$. In the figure this added differential is denoted only by a plus sign. Similarly, $q_{xz}+$ denotes $q_{xz} + \frac{\partial q_{xz}}{\partial x}dx$, $p_y +$ denotes $p_y + \left(\frac{\partial p_y}{\partial y}\right)dy$, $q_{zy}+$ denotes $q_{zy} + \frac{\partial q_{zy}}{\partial z}dz$, etc.

As the scope of this text precludes the discussion of the theory of stress in three dimensions, we shall use the above notation and Fig. 67 simple to establish a fundamental theorem, namely:

Theorem I. The shearing stresses not only across two mutually perpendicular planes but also perpendicular to the intersection of these planes are always equal in magnitude and both act either towards or away from this intersection.

To prove this theorem, consider the differential element of Fig. 67 as a free body, then as the body of which it is a part is in equilibrium it must be in equilibrium under the action of the **forces** on its faces due to the **stresses shown** in the figure and such other distributed forces as its weight, etc.

The forces due to the stresses may all be conceived to act at the center of areas of the corresponding faces, for on the differential surfaces forming these faces the stresses may be assumed constant. For similar reasons all forces such as weight, etc., may be assumed to be concentrated at the center of volume of the element.

To avoid these latter forces in writing an equation expressing a condition of equilibrium of rotation (Σ Moment

= o), let us use as axis of moments the line indicated in Fig. 67, by p_z and $p_z +$. Noting also that we now deal with **forces** and not with stresses, and that, therefore, p_z, q_{zx}, etc., must all be multiplied by the **areas of the faces** upon which they act, so that forces may be obtained, we have as the sum of the moments of all forces about the selected axis,

$$\{q_{yx} dx\, dz\}\frac{dy}{2} + \left\{\left(q_{yx} + \frac{\partial q_{yx}}{\partial y} dy\right) dx\, dz\right\}\frac{dy}{2} - \{q_{xy}\, dy\, dz\}\frac{dx}{2}$$
$$- \left\{\left(q_{xy} + \frac{\partial q_{xy}}{\partial x} dx\right) dy\, dz\right\}\frac{dx}{2} = 0.$$

Dividing by $\dfrac{dx\, dy\, dz}{2}$, we obtain

$$q_{yx} + \left(q_{yx} + \frac{\partial q_{yx}}{\partial y} dy\right) - q_{xy} - \left(q_{xy} + \frac{\partial q_{xy}}{\partial x} dx\right) = 0,$$

or
$$q_{yx} - q_{xy} = 0.$$

So that
$$\mathbf{q}_{yx} = \mathbf{q}_{xy},$$

which proves our theorem.

Exercise 148. Show that $q_{xz} = q_{zx}$ and that $q_{yz} = q_{zy}$.

Compare the above theorem with the results obtained on page 67.

Two Dimensional Stresses. — To simplify the mathematics and so more rapidly arrive at the results bearing more directly on practical considerations, let us pass to two-dimensional stress.

In Fig. 67, conceive no stresses to act across the planes perpendicular to the Z-axis. Then all stresses across plane 2 and 5 (i.e., p_z, q_{zx}, q_{zy} and $p_z +$, $q_{zx} +$, $q_{zy} +$) reduce to zero. But by Theorem I, with q_{zx} and q_{zy}, q_{xz} and q_{yz} also become zero.

The remaining stresses are shown in Fig. 68, or more simply in two dimensions in Fig. 69, where the subscripts

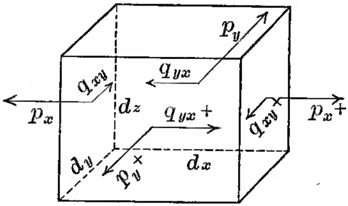


FIG. 68

to q are omitted, for $q_{xy} = q_{yx}$, and they are thus no longer needed.

EXERCISE 149. Show that a rectangular parallelopiped, $a \times b \times c$, with no stresses on the faces $b \times c$, is in equilibrium under the conditions of stress shown in Fig. 69, when the weight of the material is neglected.

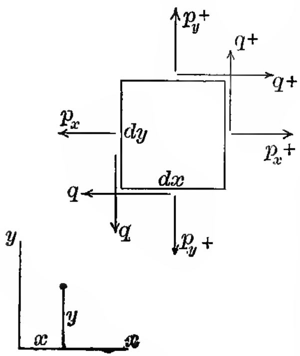


FIG. 69

Given the stresses across any two mutually perpendicular planes at any point in a stressed material, to determine the stresses across any other plane at the same point. This is the statement of a fundamental problem in the theory of stress. To fully grasp this problem, consider the point P (x, y) in Fig. 70 (a) as the point at which the stresses are to be discussed.

Imagine a plane perpendicular to the y-axis and passing through P. Then the most general condition of stress across this plane, within the stressed material, consists of a normal stress p_y and a shearing stress q, as represented in Fig. 70 (b). Moreover, let the directions of the stresses there shown be the positive directions; so that a **tensile normal stress** shall be **positive,** and a **shearing stress** which tends to move material on the **positive** (upper) **side** of the plane in a **positive direction** (toward the right) shall be **positive.**

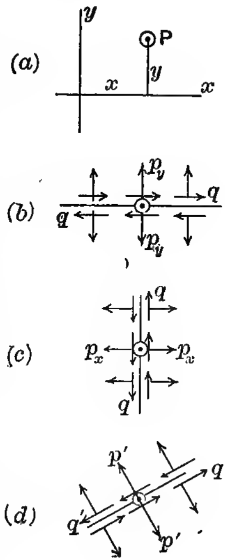


FIG. 70

Imagine now a plane perpendicular to the x-axis and passing through P. The most general condition of stress across this plane is indicated, with due regard to signs, in Fig. 70 (c).

Finally, in Fig. 70 (d) the stresses across any oblique plane are represented. The **stresses there shown are to be regarded as positive.** Any reversal of these stresses will then be indicated by a negative sign attached to either p' or q' at the conclusion of any computation.

It is to be noted that as any and all stresses indicated in Fig. 70 may vary from point to point within the material, the areas of the surfaces indicated in this and all similar figures must be regarded as infinitesimal. Under this condition the stresses may be considered constant over the surfaces considered, and the force due to any stress may be found by multiplying the stress by the area of the surface across which it acts, and this force may be regarded as concentrated at the center of said area.

To establish the relations existing between the various stresses shown in Fig. 70, make a free body of an element of the material at the point P, this element to be bounded by planes parallel to the planes shown in Figs. 70 (b), (c), and (d), and two planes parallel to the xy-plane a distance dz apart. This element is shown in Fig. 71; note that only the stresses acting on the element must be used. The stresses omitted in Fig. 71 but indicated in Fig. 70 are the stresses acting on the material which had to be removed in order to make a free body of the element considered.

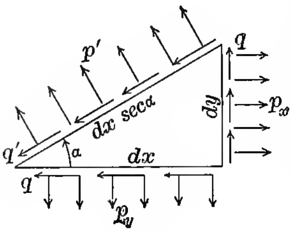


FIG. 71

In connection with Fig. 71 the problem to be solved may now be restated as follows:

Given the stresses p_x, p_y, and q, **also the angle** α, **to find the stresses p′ and q′ in terms of the given quantities.**

As the body, of which the element represented in Fig. 71 is a part, is assumed in equilibrium, the element itself is in equilibrium, and we may write the

$$\Sigma \text{ horizontal forces} = 0,$$

and the
$$\Sigma \text{ vertical forces} = 0.$$

Thus we obtain

$$-(p'\,dx\,dz \sec\alpha)\sin\alpha - (q'\,dx\,dz\sec\alpha)\cos\alpha - (q\,dx\,dz) + (p_x dy\,dz) = 0,$$

and

$$+ (p' \, dx \, dz \sec \alpha) \cos \alpha - (q' \, dx \, dz \sec \alpha) \sin \alpha + (q \, dy \, dz) - (p_y dx \, dz) = 0.$$

Placing $dy = dx \tan \alpha$ and dividing by $dx \, dz \sec \alpha$, we obtain

$$- p' \sin \alpha - q' \cos \alpha - q \cos \alpha + p_x \sin \alpha = 0,$$

and $$p' \cos \alpha - q' \sin \alpha + q \sin \alpha - p_y \cos \alpha = 0.$$

Eliminating q' (by multiplication by $\sin \alpha$ and $\cos \alpha$ respectively, and subsequent subtraction), we obtain,

$$p' = p_x \sin^2 \alpha + p_y \cos^2 \alpha - 2 \, q \sin \alpha \cos \alpha.$$

The elimination of p' gives

$$q' = (p_x - p_y) \sin \alpha \cos \alpha + q(\sin^2\alpha - \cos^2\alpha).$$

Changing all functions of α in these equations to functions of 2α, we have

$$\mathbf{p}' = \frac{\mathbf{p}_x + \mathbf{p}_y}{2} - \frac{\mathbf{p}_x - \mathbf{p}_y}{2} \cos 2 \alpha - \mathbf{q} \sin 2 \alpha, \quad . \quad \text{(A)}$$

and

$$\mathbf{q}' = \frac{\mathbf{p}_x - \mathbf{p}_y}{2} \sin 2 \alpha - \mathbf{q} \cos 2 \alpha. \quad . \quad . \quad . \quad . \quad . \quad \text{(B)}$$

Exercise 150. Obtain equations (A) and (B).

Exercise 151. A simple beam 2 inches by 10 inches, span 14 feet, carries a uniform load of 60 pounds per foot run, and a concentrated load of 200 pounds at mid-span. Find the normal stress and the shearing stress at a point 3 inches above the neutral axis and 3 feet from the left support across a plane inclined at 22½ degrees to the neutral surface. Sketch the stresses across a horizontal and a vertical plane through the above point and sketch the differential element bounded by these planes and the oblique plane. Note that under the assumption made in the study of beam stresses, normal stresses across horizontal planes do not occur.

It is seldom necessary to find the stresses across any oblique plane as in Ex. 151. It is far more important to determine the particular values of α, which locate the planes across which p' and q' reach their maximum and minimum values, and then determine these values. Again, it is important to locate the planes across which normal stresses alone act or across which shearing stresses alone act, i.e., the planes of **pure stress.**

Principal Planes and Principal Stresses. — Principal planes are the planes, passing through any given point in a stressed material, across which normal stresses **alone act.** Thus **the principal planes are planes of zero shear and planes of pure normal stress.**

To obtain the inclination of the principal planes, as referred to the assumed axes of x and y parallel to which the stresses are known, Fig. 70 (a), (b), and (c), place $q' = 0$ in equation (B),

whence
$$\tan 2\alpha = \frac{2\,\mathbf{q}}{\mathbf{p}_x - \mathbf{p}_y}, \quad . \quad . \quad . \quad . \quad . \quad (C)$$

or
$$2\alpha = \text{arc tan}\,\frac{2\,q}{p_x - p_y} + n\pi \quad (n \text{ any integer}),$$

and
$$\alpha = \tfrac{1}{2}\,\text{arc tan}\,\frac{2\,q}{p_x - p_y} + n\,\frac{\pi}{2}\,.$$

Thus the values of α which determine the directions of the principal planes differ by $\frac{\pi}{2}$ or 90°, or

Theorem II. The principal planes are mutually perpendicular.

Planes of Maximum and Minimum Normal Stress. — To locate these planes it is necessary to determine the values of α which will make p' as given in equation (A) a maximum or a minimum.

The derivative of

$$p' = \frac{p_x + p_y}{2} - \frac{p_x - p_y}{2}\cos 2\alpha - q \sin 2\alpha,$$

where p_x, p_y, and q are constants, with respect to α, is

$$\frac{dp'}{d\alpha} = + (p_x - p_y)\sin 2\alpha - 2q\cos 2\alpha.$$

Placing this derivative equal to zero and solving for α, we obtain

$$\tan 2\alpha = \frac{2q}{p_x - p_y},$$

which is the same as equation (C); therefore,

Theorem III. The principal planes are also planes of maximum or minimum normal stress.

Principal stresses are the normal stresses acting across the principal planes. They **are pure normal stresses,** for the shearing stresses on the principal planes are, by definition, equal to zero.

The magnitude of the principal stresses is obtained by substituting the value of α determined by equation (C) into equation (A). From $\tan 2\alpha = \frac{2q}{p_x - p_y}$ it follows that

$$\sin 2\alpha = \frac{2q}{\sqrt{4q^2 + (p_x - p_y)^2}}$$

and

$$\cos 2\alpha = \frac{p_x - p_y}{\sqrt{4q^2 + (p_x - p_y)^2}},$$

whence the **principal stresses** as determined from equation (A) are

$$\left.\begin{aligned} \mathbf{p}_1 &= \frac{\mathbf{p}_x + \mathbf{p}_y}{2} + \frac{1}{2}\sqrt{4\mathbf{q}^2 + (\mathbf{p}_x - \mathbf{p}_y)^2}, \\ &\text{and} \\ \mathbf{p}_2 &= \frac{\mathbf{p}_x + \mathbf{p}_y}{2} - \frac{1}{2}\sqrt{4\mathbf{q}^2 + (\mathbf{p}_x - \mathbf{p}_y)^2}. \end{aligned}\right\} \quad \text{(D)}$$

It should also be remembered that p_1 is the maximum and that p_2 is the minimum value of p' in equation (A). By substituting in $\frac{d^2p'}{d\alpha^2}$ the various values of α determined by equation (C), the planes of maximum and minimum normal stress are readily distinguished.

EXERCISE 152. Show, by substituting the value of α determined by equation (C) in equation (B), that the shearing stresses on the planes of maximum and minimum normal stress are equal to zero.

EXERCISE 153. Find the principal stresses and the principal planes at the point and in the beam described in Ex. 151.

Sketch these planes and stresses.

Surfaces of Principal Stress. — In any stressed material surfaces across which pure normal stresses act may be traced. These surfaces may be divided into two sets; across one set the stress will be tensile, across the other it will be compressive. By reason of Theorem II, these sets of surfaces will intersect at right angles.

As an illustration, trace the surfaces of principal stress for a uniformly loaded simple beam, Fig. 72.

The normal stress across vertical planes is given by the equation $p = \frac{My}{I}$ and the normal stress across horizontal planes is assumed to be zero. The shearing stress across horizontal and vertical planes is given by $q = \frac{Q_x}{Ib} yA.$

Assuming the axis of x horizontal, then in the notation of equations (A), (B), (C), and (D), $p_x = p$, $p_y = 0$, and $q = q$; so that equation (C), determining the directions of the principal planes, becomes

$$\tan 2\alpha = \frac{2q}{p}.$$

Along the plane AB, Fig. 72, $q = 0$, therefore $\alpha = 0°$ or $90°$; evidently the planes inclined at $90°$ to

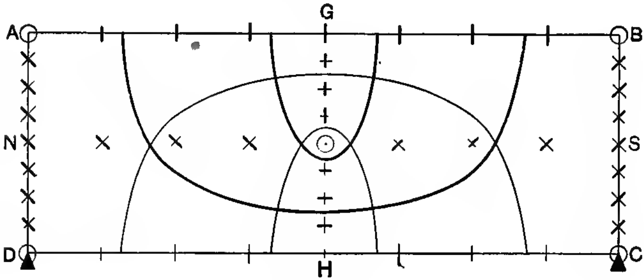


FIG. 72

the axis of x are the planes of pure compression indicated by heavy lines.

At A and B both $q = 0$ and $p = 0$, therefore α is indeterminate; $p = 0$ because on the section AD, $M = 0$.

Similarly, along the plane DC the lightly drawn lines indicate the planes of pure tension.

Along AD, $p = 0$ (for $M = 0$) and q is zero only at A and D, therefore $\alpha = 45°$ or $135°$. Here again the heavy lines indicate compression planes and the light lines tension planes.

Similarly, along BC, as shown in Fig. 72.

Along the section GH we have $q = 0$ (for $Q_x = 0$) and p is zero only on NS. Hence the intersection of GH and NS is an indeterminate point, while at other points in the section $\alpha = 0°$ and $90°$.

Along NS, $p = 0$, therefore $\alpha = 45^\circ$ or 135°, as shown.

Following the general directions obtained above, the heavy lines indicate surfaces of pure compression and the light lines indicate surfaces of pure tension.

As an application of these orthogonal cylindrical surfaces, attention is called to the steel reinforcements in concrete beams. As is well known, concrete is strong in compression but weak in tension; the reinforcement is introduced to carry tension. The steel rod should thus be normal to the surfaces of pure tension or they should follow the surfaces of pure compression.

EXERCISE 154. Sketch an effective arrangement of reinforcement by means of straight rods for the beam shown in Fig. 72. Why should no reinforcing rods be introduced along the plane GH, Fig. 72?

EXERCISE 155. Sketch roughly the surfaces of maximum and minimum shearing stress for the beam shown in Fig. 72.

Maximum Shear. — To investigate the conditions leading to maximum shear, use equation (B), namely,

$$q' = \frac{p_x - p_y}{2} \sin 2\alpha - q \cos 2\alpha,$$

whence
$$\frac{dq'}{d\alpha} = (p_x - p_y) \cos 2\alpha + 2q \sin 2\alpha.$$

For maxima and minima conditions, place this equal to zero and solve for α. Thus we find

$$\tan 2\alpha = -\frac{p_x - p_y}{2q}.$$

This is evidently the negative reciprocal of the value of $\tan 2\alpha$ as given in equation (C).

And as angles whose tangents are negative reciprocals differ by 90°, we may state that the **double** of the angles determining the principal planes differs from the double of the angles determining the planes of maximum shear by 90°. Therefore,

Theorem IV. The planes of maximum and minimum shear bisect the angles between the principal planes.

EXERCISE 156. Compute the maximum and the minimum shearing stresses.

EXERCISE 157. Compute the maximum and the minimum normal stresses.

The maximum and minimum shearing stresses are readily shown to be

$$q'_{\substack{\max \\ \min}} = \pm \tfrac{1}{2} \sqrt{4\,q^2 + (p_x - p_y)^2} \quad . \quad . \quad . \quad \text{(E)}$$

EXERCISE 158. Find the normal stress on the planes of maximum and minimum shear.

Note, from Ex. 158, that **the normal stresses on the planes of maximum and minimum shear are not zero,** so that these planes are not planes of pure shear.

EXERCISE 159. Find the values of the maximum and the minimum shearing stresses and their planes for the point described in Ex. 151.

Sketch these planes and stresses.

Pure Shear. — Planes across which the only stress is a shearing stress are called planes of pure shear.

To arrive at the conditions leading to pure shear, p' in equation (A), p. 139, must be placed equal to zero, and the resulting equation solved for α. This value of

α when substituted in equation (B) gives the shearing stresses on the planes of pure shear.

EXERCISE 160. Solve equation (A) for α.

Instead of performing the operations indicated above and thus locating the plane of pure shear with reference to an arbitrarily selected axis of x, as shown in Figs. 70 and 71, it will be found more convenient first to locate the principal planes by means of equation (C) and then to locate the required planes with reference to these principal planes.

After α has been determined by means of

$$\tan 2\alpha = \frac{2q}{p_x - p_y},$$

a new element, at the point considered, having its bounding planes parallel to the directions determined

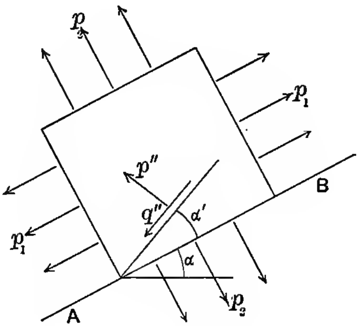


FIG. 73

by α, may be isolated. The normal stresses on this element, as indicated in Fig. 73, are found by means of equations (D); the shearing stresses are zero.

The stresses p'' and q'' across any plane inclined at an angle α' to the direction AB can be found by substituting p_1 for p_x, p_2 for p_y, 0 for q, and α' for α in equations (A) and (B), p. 139.

Thus $$p'' = \frac{p_1 + p_2}{2} - \frac{p_1 - p_2}{2} \cos 2\alpha',$$

and $$q'' = \frac{p_1 - p_2}{2} \sin 2\alpha'.$$

EXERCISE 161.* Derive the above equations from an element in equilibrium.

To find the angles α' which will locate the planes of pure shear with reference to the axis AB (Fig. 73), put $p'' = 0$, whence

$$\cos 2\alpha' = \frac{p_1 + p_2}{p_1 - p_2},$$

which gives the required values.

A substitution of these values of α' into the general equation for q'' yields

$$q'' = \frac{p_1 - p_2}{2}\sqrt{1 - \left(\frac{p_1 + p_2}{p_1 - p_2}\right)^2},$$

Or $$q'' = \sqrt{-p_1 p_2},$$

as the shear on the planes of pure shear.

This result calls attention to the fact that either **p_1 or p_2 must be negative** so that q'' may be real.

Or

Theorem V. Pure shear can only exist when one of the principal stresses is a compressive and the other a tensile stress.

As a special case, when the principal stresses are equal tensions and compressions the pure shear occurs on planes inclined at 45° and 135° to the principal planes,

and the shearing stresses are then equal in magnitude to the principal stresses.

EXERCISE 162. Prove the special case under Theorem V.

EXERCISE 163. If a given material is in pure shear, show that pure compressive and tensile stresses (equal in magnitude to the shearing stress) act across planes inclined at 45° and 135° to the planes of pure shear.

The important facts stated in Theorem V and Ex. 163 can best be visualized as shown in Figs. 74 and 75. Here all squares represent differential elements of the material at the point considered; they are drawn of different sizes simply for convenience of representation.

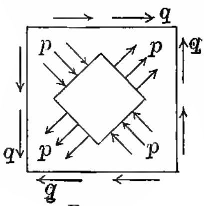


FIG. 74

In Fig. 74 the horizontal and vertical planes are planes of pure shear; the planes inclined at 45° to these become planes of pure normal stress.

If the horizontal and vertical planes are planes of **pure normal stress, one a tension, the other an equal compression,** then the **planes inclined at** 45° to these **are planes of pure shear,** and the **shearing stresses are equal in magnitude** to the normal stresses on the other planes.

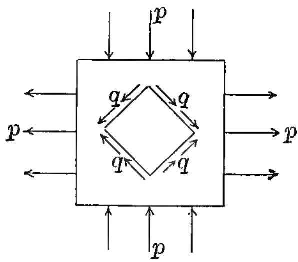


FIG. 75

As an illustration of this state of stress, shafts in torsion may be cited. The material at the cylindrical surface of such shafts is in pure shear across planes normal to the axis and across planes passing through the axis of the shaft, as shown in Fig.

76. But across planes passing through the intersection of these planes and inclined to them at 45° the material is in **tension** (along AB) and in compression (along CD). If the shaft is composed of material weaker in tension than in shear, the shaft fails, not in shear along DB,

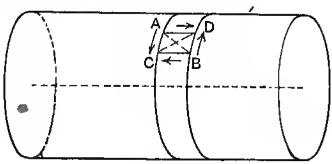


FIG. 76

but in tension along AB. Apply torsion to a blackboard crayon and note the line of fracture along the surface of the crayon, as an illustration of this fact.

EXERCISE 164. A steel shaft 2 inches in diameter transmits 26 H.P. at 90 r.p.m. Find its factor of safety for shear and for tensile stress.

EXERCISE 165. Investigate the general condition of a material under the action of two equal tensile stresses at right angles, and state your result in the form of a theorem.

Linear Stress. — If in Fig. 67 all stresses across planes perpendicular to the y-axis as well as all stresses across planes perpendicular to the z-axis are considered equal to zero, then the only stress still acting is p_x. Material under the action of a single normal stress is said to be subject to a linear stress. A tension or compression specimen under test is in linear stress.

EXERCISE 166. Show, by means of equations (A) and (B), p. 139, that the maximum shearing stress in a material under compressive linear stress acts across planes inclined at 45° to the planes of pure compression and is equal in magnitude to one-half the compressive stress. Is this a pure shear?

EXERCISE 167. A wrought-iron bar $\frac{5}{8}$ of an inch by 4 inches is under a tension of 18,000 pounds. Find the maximum unit shear in the bar.

EXERCISE 168. A steel block 2 inches square and 3 inches long is under a compression of 200,000 pounds. Find the factor of safety for shear and for compression.

In compression tests on wood and brittle material, such as stone and cast iron, failure occurs by shearing. The angle between the plane of rupture and the axis of the stress is, however, never 45°, but always less than 45°.

In a tension test on a polished specimen of mild steel, lines forming rough helices about the cylindrical surface of the test piece are clearly visible after the yield point has been reached. These lines, known as **Luders' Lines,** are inclined at about 60° to the axis of stress and indicate molecular slip due to a shearing action.

The difference in location between the observed planes of rupture and computed location of the planes of maximum shearing stress (Ex. 166) may be accounted for in some measure by the fact that the internal resistance to sliding, due to the normal stresses and the cohesive force of the molecules, has been neglected in these computations.

Internal Resistance to Sliding. — In an element of a material in tensile linear stress, as shown in Fig. 77, the following relations are easily established from the conditions of equilibrium:

$$p' = p_x \sin^2 \alpha,$$
$$q' = p_x \sin \alpha \cos \alpha.$$

If now μ is the coefficient of internal resistance to slipping, and c is the normal cohesion per unit of area, then when slipping along the plane of rupture is about to occur we have

$$\mu = \frac{\text{shearing stress}}{\text{normal pressure}} = \frac{p_x \sin \alpha \cos \alpha}{c - p_x \sin^2 \alpha},$$

for evidently the normal stress, $p' = p_x \sin^2 \alpha$, tends to reduce the cohesive force between the molecules.

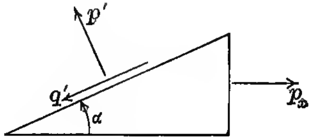


FIG. 77

The above equation when solved for p_x yields

$$p_x = \frac{\mu c}{\mu \sin^2 \alpha + \sin \alpha \cos \alpha},$$

the tensile stress which will just overcome the internal resistance along a plane inclined at an angle α to the axis of stress.

Rupture will occur along a plane whose α is such as to cause p_x to be a minimum or which causes

$$\mu \sin^2 \alpha + \sin \alpha \cos \alpha$$

to become a maximum. This value of α is determined by

$$\cot 2\alpha = -\mu.$$

EXERCISE 169. Find the value of α for which rupture occurs under compression. What relation does this angle bear to the angle of rupture for tension?

EXERCISE 170. In a compression test of cast iron the plane of rupture was inclined at an angle of 35° to the axis of stress. Compute μ.

EXERCISE 171. If the ultimate strength in compression of the cast iron in Ex. 170 was 90,000 pounds per square inch, what was the cohesive force between the molecules in pounds per square inch?

EXERCISE 172. Compute the probable tensile strength of cast iron by the above formulas. Does this agree with the average experimental results?

The result of Ex. 172 calls attention to the fact that the above theory is by no means complete. Either c is not the same for compression as for tension or other considerations have been entirely neglected.

Ellipse of Stress. — Let p_1 and p_2 be the principal stresses at any point in a stressed material. Then from equations (A) and (B), p. 139, if p_x, p_y, and q are replaced by p_1, p_2, and zero, respectively, we obtain

$$p' = p_1 \sin^2 \alpha + p_2 \cos^2 \alpha,$$

and

$$q' = (p_1 - p_2) \sin \alpha \cos \alpha,$$

where p' and q' now represent the normal and the shearing stress, respectively, across any plane inclined at an angle α to the direction of the stress p_1.

The resultant (oblique) stress across this plane is thus

$$S' = \sqrt{p'^2 + q'^2} = \sqrt{p_1{}^2 \sin^2 \alpha + p_2 \cos^2 \alpha},$$

for the stresses S', p', and q' are all distributed over the same area.

Fig. 78 shows the directions of these stresses.

EXERCISE 173. Obtain the above results from the equilibrium of any element of the material considered without referring to equations previously obtained.

EXERCISE 174. Find the x and y components of the resultant stress S' directly from the values of p' and q' in terms of p_1, p_2, and α.

The x and y components of S' as obtained from Ex. 174 are $p_x' = -p_1 \sin \alpha$ and $p_y' = -p_2 \cos \alpha$.

If these values are considered as the x and y coördinates of the end P of the vector S' representing the resultant stress across the plane AB, Fig. 78, so that

$$x = -p_1 \sin \alpha \quad \text{and} \quad y = -p_2 \cos \alpha,$$

we have $$\left(\frac{x}{p_1}\right)^2 + \left(\frac{y}{p_2}\right)^2 = \sin^2 \alpha + \cos^2 \alpha = 1.$$

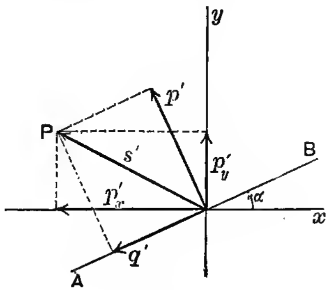


FIG. 78

Therefore, the locus of P for all inclinations of the plane AB is an ellipse whose center is at the point at which the stresses are considered and whose semi-axes are equal in magnitude to and coincide in direction with the principal stresses at this point.

From the above discussion **a geometrical construction for obtaining the resultant stress on any plane** through any point, provided the principal stresses at this point are known, is readily obtained.

Fig. 79 illustrates the construction.

EXERCISE 175. Show that the construction illustrated in Fig. 79 gives S, the resultant stress on the plane AB in direction and magnitude.

EXERCISE 176. If one of the principal stresses, say p_1, parallel to the x-axis becomes negative, i.e., changes to a compression, what change in Fig. 79 will this necessitate?

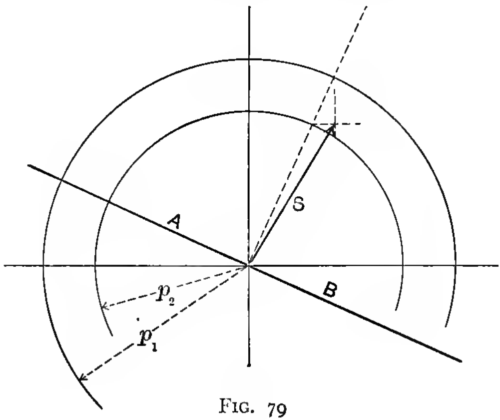


FIG. 79

SECTION XVI

STRAIN

The changes in the form or the dimensions of a body due to the action of external forces (or the stresses induced by them) are usually called strains. Thus, strains are the changes in the relative positions of points in a given material due to the stresses within this material.

As already stated on p. 4, it is more convenient to use the term strain in a narrower sense. The word **strain will be used to denote the change in the relative position of two points originally at unit distance from each other,** or, what amounts to the same thing, the total change in the relative position of any two points divided by the unstrained distance between these points.

This **strain per unit length** is sometimes called the unit strain.

Strains Due to Normal Stresses. — Normal stress always produces a change in the linear dimensions of the material. Whenever a single normal stress acts it produces a change not only in the dimension parallel to, but also in all dimensions perpendicular to, the direction of the stress.

In Fig. 80 a tensile stress, p, in the direction AB will not only lengthen the dimension AB of the element, but

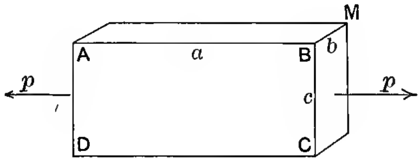


FIG. 80

will also shorten the dimensions BM and BC of this element.

If AB has an unstrained length, a, then the **longitudinal strain** (in the direction of the stress) may be represented by $\frac{\Delta a}{a}$.

Similarly, if the unstrained lengths of BM and BC are b and c respectively, then the **lateral strains** may be represented by $\frac{\Delta b}{b}$ and $\frac{\Delta c}{c}$.

Experiment shows

(a) that the lateral strains are equal, i.e.,

$$\frac{\Delta b}{b} = \frac{\Delta c}{c};$$

(b) that for a given material and within the elastic limit the ratio

$$\frac{\text{lateral strain}}{\text{longitudinal strain}} \text{ is constant.}$$

This ratio is called **Poisson's Ratio** and is usually denoted by $\frac{1}{m}$. Thus,

$$(\text{lateral strain}) = \frac{1}{m} (\text{longitudinal strain})$$

when the strains are due to a single normal stress.

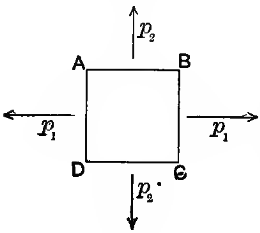


FIG. 81

If **two principal stresses** act simultaneously, then each produces both its longitudinal and lateral strains. Thus if s_1' represents the **actual strain** in the direction AB (Fig. 81) and s_2' represents the actual strain in the direction BC, while s_1 represents the longitudinal strain in the direction AB due to p_1 **acting alone,** and s_2 is the longitudinal strain due to p_2 **acting alone,** we have the strain

	in the direction of	
	p_1	p_2
due to p_1	s_1	$-\frac{s_1}{m}$
due to p_2	$-\frac{s_2}{m}$	s_2

whence $$s_1' = s_1 - \frac{s_2}{m}$$

and $$s_2' = s_2 - \frac{s_1}{m}.$$

EXERCISE 177. If s_1, s_2, and s_3 are the longitudinal strains due to the three mutually perpendicular stresses, each acting separately, find the actual strains in the directions of the given stresses.

Strains Due to Shearing Stresses. — Shearing stresses do not change the dimensions of the stressed material, but change its form. Shearing stresses will deform the square $ABCD$, Fig. 82, into the rhombus $A'B'C'D'$. The relative change in position of the points $ABCD$ can here best be expressed as the change of the angle ADC $\left(=\frac{\pi}{2}\text{radians}\right)$ to

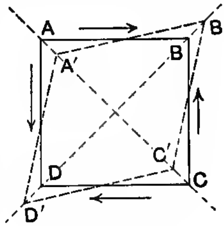


FIG. 82

the angle $A'D'C'$ $\left(=\frac{\pi}{2}-\phi\text{ radians}\right)$. This **change in the angle ADC,** namely, ϕ radians, is independent of the linear dimensions of the element and may thus be regarded as a strain, the **shearing strain.**

EXERCISE 178. Show that as ϕ is always a small angle, the actual strain along the diagonals BD and AC in Fig. 82 is $\frac{\phi}{2}$.

Note that although the edges of the element in **pure shear,** in Fig. 82, do not change in length under this shearing stress, the diagonals of this element do change in length. This change in length is not due to the shearing stress, but to the tensile and compressive stresses equal in magnitude to the shearing stresses which always accompany the shears (see Ex. 163).

Volumetric Strain. — Volumetric strain is the change in volume due to hydrostatic stress divided by the unstrained volume.

Hydrostatic stress is a stress due to three equal mutually perpendicular normal stresses. It derives its name from the analogy to fluid pressure.

If x is the unstrained length of the edge of a cubical element and this edge changes in length by Δx under hydrostatic stress, then the volumetric strain is

$$\frac{x^3 - (x - \Delta x)^3}{x^3} = \frac{+ 3\,x^2\Delta x - 3\,x(\Delta x)^2 + (\Delta x)^3}{x^3},$$

and as Δx is very small in comparison to x, the terms involving $(\Delta x)^2$ and $(\Delta x)^3$ may be neglected; whence

$$\text{the volumetric strain} = \frac{3\,\Delta x}{x}.$$

The Relation between Stress and Strain. — It has been demonstrated experimentally that the ratio of a stress to the strain it produces is always constant for a given material, provided the stress does not exceed a certain limit called the **elastic limit.**

Thus:

$$\frac{\text{Stress}}{\text{Strain}} = \text{constant.}$$

The ratio of a stress to its strain is called a modulus of elasticity.

Three Important Moduli of Elasticity. — **Young's Modulus of Elasticity,** denoted by **E**, is the modulus for **pure normal stress in one direction only.** It is found experimentally by testing bars in tension or compression. If p represents a stress in pounds per square inch within the elastic limit of the material and s represents the corresponding strain in inches per inch, then

$$\mathbf{E} = \frac{\mathbf{p}}{\mathbf{s}} \text{ pounds per square inch.}$$

The value for E is practically the same for compressive and tensile stresses.

The **Modulus of Rigidity** or the **Shearing Modulus of Elasticity,** denoted by **G,** is the modulus for **pure shearing stress.** If q represents a shearing stress in pounds per square inch within the elastic limit of the material and ϕ represents the corresponding strain in radians, then

$$\mathrm{G} = \frac{q}{\phi} \text{ pounds per square inch.}$$

The Bulk Modulus, denoted by **K,** is the modulus for hydrostatic stress (resulting from three mutually perpendicular and equal normal stresses). If p represents a hydrostatic stress in pounds per square inch within the elastic limit of the material and x represents the unstrained length of the edge of a cube to be submitted to this stress, then the volumetric strain is $\frac{3\,\Delta x}{x}$ (see page 158),

and
$$\mathrm{K} = \frac{\textbf{hydrostatic stress}}{\textbf{volumetric strain}} = \frac{px}{3\,\Delta x}.$$

Relations between the Elastic Constants

Relation between E, G, and $\frac{1}{m}$. — As E is the modulus of pure normal stress in one direction only, and G is the modulus of pure shear, they can only be compared when the material considered is under both pure normal stress and pure shear across different planes.

Rereading of Theorem V, page 147, will show that these conditions are satisfied whenever the material is under equal tensile and compressive stresses across mutually perpendicular planes.

Fig. 83 represents an element under these conditions. Let x be the original unstrained length of the edges of this element. Then as the material is considered iso-

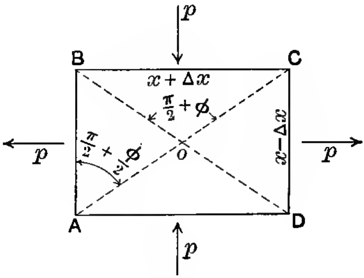


FIG. 83

tropic, and as the tension equals the compression, the strained dimensions are $x + \Delta x$ and $x - \Delta x$. Also, the angle BOC, originally $\frac{\pi}{2}$ radians, becomes $\frac{\pi}{2} + \phi$ radians under the action of the pure shearing stresses, q, acting across the planes BD and AC.

The angle BAC is evidently $\frac{\pi}{4} + \frac{\phi}{2}$ radians.

E may readily be expressed in terms of $\Delta x, x, \frac{1}{m}$, and p; for the actual strain in the direction BC is

$$\frac{\Delta x}{x} = \frac{p}{E} + \frac{1}{m}\frac{p}{E} = \frac{p}{E}\left(1 + \frac{1}{m}\right) \text{ (see page 156).}$$

Also
$$G = \frac{q}{\phi},$$

and as from ΔABC

$$\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \frac{x + \Delta x}{x - \Delta x},$$

or $$\frac{\tan\frac{\pi}{4} + \tan\frac{\phi}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\phi}{2}} = \frac{1 + \frac{\phi}{2}}{1 - \frac{\phi}{2}} = \frac{x + \Delta x}{x - \Delta x},$$

whence $$\frac{\Delta x}{x} = \frac{\phi}{2},$$

we have $$\frac{\Delta x}{x} = \frac{q}{2G}.$$

Thus $$\frac{\Delta x}{x} = \frac{q}{2G} = \frac{p}{E}\left(1 + \frac{1}{m}\right),$$

and as $$p = q \quad \text{(see page 148)},$$

we have $$\frac{\mathbf{E}}{\mathbf{G}} = 2\left(1 + \frac{1}{\mathbf{m}}\right).$$

Exercise 179. Obtain this same relation by the use of the facts expressed in Exs. 163 and 178.

Exercise 180. Find the value of $\frac{1}{m}$ for steel from the values for E and G given on page 9.

Exercise 181. What is the value of $\frac{1}{m}$ for cast iron?

Exercise 182. By considering a cubical element under the action of hydrostatic stress, show that

$$\mathbf{E} = 3\mathbf{K}\left(1 - \frac{2}{\mathbf{m}}\right).$$

Exercise 183. Show that

$$\mathbf{E} = \frac{9\mathbf{KE}}{\mathbf{G} + 3\mathbf{K}}.$$

Section XVII

ELASTIC FAILURE

In all preceding problems elastic failure has been considered imminent when the stress in the material considered has reached a certain working stress; this working stress being the ultimate strength divided by a suitable factor of safety (see page 10). This method does very well when only one stress (shearing or normal) occurs on the material considered.

If, however, two principal stresses occur simultaneously at any point in the material, it may well be questioned whether the effect of the greater of these principal stresses should be considered alone or whether the combined effect of all the principal stresses is the cause of elastic failure.*

We shall consider only three theories as to the manner in which failure may occur.

Elastic failure may occur according to
theory I, for a given value of **the greatest principal stress,**
theory II, for a given value of **the maximum shear,**
theory III, for a given value of **the greatest actual strain.**

According to **theory I,** the principal stresses at the point in the material under investigation should be determined by equations (D), page 141. These are pure normal stresses, so that shear need not be considered. The greater of these principal stresses should then be less than the safe working stress of the material for tension or compression, as the case may be, in order that the structure may be safe.

Note that under this theory the effect of the lesser principal stress is neglected, the material being considered as under the action of one normal stress only. For this reason this theory is hardly complete, although

* For further information on this subject consult C. Bach, "Elastizität und Festigkeit," Fünfte Auflage, 1905, pp. 424–428; A. Föppl, "Technische Mechanik," 1907, Fünfter Band, p. 19; also papers by J. J. Guest, Phil., May, July, 1900, and Mohr, Zeitschr. d. V. D. Ing. 1900, s. 1524.

when used with a factor of safety varying with the conditions of the compound stress it has given satisfactory results.

Theory II calls for the use of the maximum shear as the determining factor for elastic failure. This theory has been demonstrated experimentally for ductile materials, especially mild steel. Under this theory the maximum shear, as computed from equation (E), page 145, should remain less than the safe working stress for shear for the material considered, in order to insure a safe structure.

Theory III assumes elastic failure under the greatest actual strain. That is, it considers the **failure to occur** not as the result of a certain stress in pounds per square inch, but **as the result of a certain strain in inches per inch.** In this manner it is possible to take into account **the effect of both principal stresses.**

If the material is under the action of two principal stresses p_1 and p_2 (Fig. 81), then the actual strains are

$$s_1' = s_1 - \frac{s_2}{m},$$

in the direction of the stress p_1,

and $$s_2' = s_2 - \frac{s_1}{m},$$

in the direction of the stress p_2 (see page 156).

Here $s_1 = \dfrac{p_1}{E}$ and $s_2 = \dfrac{p_2}{E}$, for these are the longitudinal stresses due to p_1 and p_2 respectively; so that

$$s_1' = \frac{p_1}{E} - \frac{1}{m}\frac{p_2}{E}$$

and $$s_2' = \frac{p_2}{E} - \frac{1}{m}\frac{p_1}{E}$$

are the **actual strains in the directions of the principal stresses.**

It is inconvenient to deal with strains, therefore these strains are reduced to equivalent simple stresses.

An **equivalent simple stress** is a stress which, acting alone and in the direction of one of the principal stresses, would produce the actual strain in this direction.

As $p = sE$ for simple stresses, the equivalent simple stresses are

$$\mathbf{{}_1p_\epsilon = p_1 - \frac{1}{m}p_2}$$

and

$$\mathbf{{}_2p_\epsilon = p_2 - \frac{1}{m}p_1}.$$

Note again that here ${}_2p_\epsilon$ acting alone in the direction of p_2 will produce the strain in this direction due to **both** of the principal stresses p_1 and p_2 acting together.

Exercise 184. Under what conditions will an equivalent simple stress be (a) greater, (b) less than the greater of the principal stresses which it replaces?

Exercise 185. In a boiler plate the tension across a plane perpendicular to the axis of the shell is 5000 pounds per square inch, and across a plane through the axis of the shell the tension is 10,000 pounds per square inch. Find the greater of the equivalent simple stresses if Poisson's Ratio is $\frac{1}{4}$.

Exercise 186. Find the three equivalent simple stresses replacing two pure tensions p_1 and p_2 across mutually perpendicular planes and a pure compression p_3 across a plane perpendicular to both the other planes.

Elastic Failure in Beams. — In beams, under the assumptions that plane sections normal to the axis of the beam remain plane sections, that the horizontal fibers act independently of each other and follow Hooke's Law, and finally, that no normal stresses are transmitted across horizontal planes, the points under the action of compound stresses need not be investigated; for the greatest tensile stress occurs on the extreme upper fibers, and the greatest compressive stress occurs on the extreme lower fibers or vice versa. In all cases the shear along these fibers is zero. Again, the shear reaches its maximum values along the neutral surface. Here again the normal stresses are zero.

In general, however, the material of the beam is under both normal stress and shear, but neither of these stresses is at its greatest value.

Thus the discussion concerning the strength of beams as already given may be considered as sufficiently extensive to cover all cases.

CHAPTER VIII

COMPOUND STRESSES

Section XVIII

COMBINED TORSION AND BENDING

When bending and torsion occur together some points in the stressed material are under compound stresses. At such points normal stresses and shearing stresses occur simultaneously. This is true not only at some

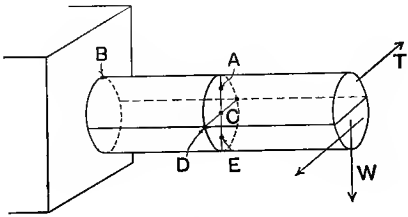


Fig. 84

points, but even at the points in the material at which the greatest stresses occur. So that in material under combined torsion and bending the effect of these compound stresses must be considered.

Fig. 84 illustrates a cylindrical bar, subjected to both the twisting action of a couple, whose moment is T, and the bending action of the force W. T produces a pure shear across all planes normal to the axis of the bar.

W produces pure normal stresses across all planes normal to the axis and pure shear across all horizontal planes.

Fig. 85 shows the stresses acting on a differential element situated at A, Fig. 84. The subscript b denotes stresses due to bending, and the subscript t the stresses due to torsion. Note that the shearing stresses always occur in pairs in accordance with Theorem I (page 134).

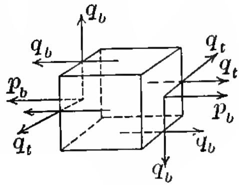


FIG. 85

EXERCISE 187. Show the stresses acting on an element at (a) C, (b) D, (c) E, in Fig. 84.

The points in the material of the bar subject to the greatest stresses are evidently at B or the diametrically opposite point, where both the normal stress due to bending and the shearing stresses due to torsion reach their greatest values. These stresses are shown in Fig. 86, and their values are

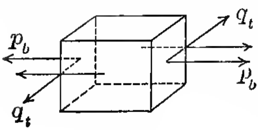


FIG. 86

$$p_b = \frac{Mr}{\frac{\pi r^4}{4}} = \frac{4\,M}{\pi r^3},$$

and

$$q_t = \frac{Tr}{\frac{\pi r^4}{2}} = \frac{2\,T}{\pi r^3},$$

where r is the radius of the cylindrical bar,

M the greatest bending moment,

and T the twisting moment acting upon the bar.

The principal stresses at the dangerous point must now be found by means of equations (D) (page 141).

Thus $$p_1 = \frac{1}{2}\left\{p_b + \sqrt{4q_t^2 + p_b^2}\right\},$$

and $$p_2 = \frac{1}{2}\left\{p_b - \sqrt{4q_t^2 + p_b^2}\right\}.$$

If elastic failure is assumed to occur under theory I (page 162), then the greater of these stresses is the dangerous stress.

Equation (E) (page 145) gives as greatest shear at the dangerous point

$$q'_{\substack{\max \\ \min}} = \pm \frac{1}{2}\sqrt{4q_t^2 + p_b^2}.$$

This according to theory II of elastic failure is the dangerous stress.

Finally, if theory III is followed, the actual strains in the direction of the principal stresses are

$$s_1' = \frac{p_1}{E} - \frac{1}{m}\frac{p_2}{E}$$

and $$s_2' = \frac{p_2}{E} - \frac{1}{m}\frac{p_1}{E},$$

or, what is more convenient, the corresponding equivalent simple stresses are

$$_1p_\epsilon = p_1 - \frac{1}{m}p_2 = \frac{1}{2}\left(1 - \frac{1}{m}\right)p_b + \frac{1}{2}\left(1 + \frac{1}{m}\right)\sqrt{4q_t^2 + p_b^2},$$

and

$$_2p_\epsilon = p_2 - \frac{1}{m}p_1 = \frac{1}{2}\left(1 - \frac{1}{m}\right)p_b - \frac{1}{2}\left(1 + \frac{1}{m}\right)\sqrt{4q_t^2 + p_b^2}.$$

According to this theory the greater of these equivalent simple stresses is the dangerous stress in the bar.

The Cranked Shaft. — In Fig. 87 the vertical force P is applied to the crank pin at a distance d_1 from the axis of the shaft. The crank is here shown in a horizontal plane. Let B represent the bearing and let d_2 be the distance from a plane through P and normal to the axis of the shaft to the outer edge of the bearing. Then by introducing the two dotted forces equal to P the problem remains unchanged, but the marked (=) forces evidently form the couple which exerts a twisting mo-

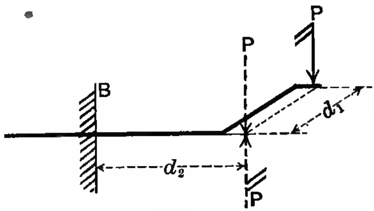


FIG. 87

ment $T = Pd_1$ upon the shaft and the remaining (dotted) force P is the force causing the bending moment $M = Pd_2$ in the shaft at B.

EXERCISE 188. Show that the *principal stresses* in the shaft, Fig. 87, at the dangerous point are

$$\frac{2}{\pi r^3}\{M \pm \sqrt{T^2 + M^2}\}.$$

EXERCISE 189. Show that the *greatest shear* at the dangerous point in the shaft, Fig. 87, is

$$\frac{2}{\pi r^3}\sqrt{T^2 + M^2}.$$

EXERCISE 190. Show that the *equivalent simple stresses* at the dangerous point in the shaft, Fig. 87, are

$$\frac{2}{\pi r^3}\left\{\left(1 - \frac{1}{m}\right)M \pm \left(1 + \frac{1}{m}\right)\sqrt{T^2 + M^2}\right\},$$

and that these reduce to

$$\frac{2}{\pi r^3}\left\{.7\,M \pm 1.3\,\sqrt{T^2 + M^2}\right\},$$

if $\frac{1}{m} = .3$, the value for steel and wrought iron.

EXERCISE 191. A steel shaft 5 inches in diameter is driven by a crank 12 inches long, and the center of the crank pin is 11 inches from the plane of the outer edge of the journal. The thrust on the crank pin normal to the plane of the crank and shaft is 10 tons. Find (a) the greatest shear, neglecting the bending; (b) the greatest normal stress, neglecting the twisting; (c) the principal stresses at the dangerous point; (d) the greatest shear at the dangerous point; (e) the equivalent simple stresses at the dangerous point.

SECTION XIX

ENVELOPES

In this section will be considered the stresses in hollow cylinders and spheres subject to fluid pressure. The weight of the fluid and the weight of the envelopes will be neglected.

When the thickness of the metal of the envelope is small compared to its other dimensions, the envelope is usually called a **shell.**

Cylindrical Shells under Internal Fluid Pressure. — Cylindrical shells subjected to internal fluid pressure, and so long that the pressure on the ends may be assumed to exert no stress on the shell, or better, cylindrical shells fitted with pistons at both ends, so arranged as to take up the end pressures independently of the shell, will first be considered. In such shells no stresses exist across planes perpendicular to their axes. The normal stresses across planes through the axes may be assumed uniformly distributed.

In Fig. 88 a quarter of a hoop of such a shell is shown as a free body. The only forces acting upon this portion of the hoop are those due to the **hoop stresses,** p_h, on the surfaces AB and CD, and those due to the internal fluid pressure w on the internal cylindrical surface.

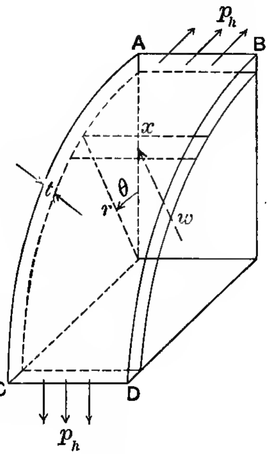


FIG. 88

As this portion of the shell is in equilibrium under the action of these forces, we may equate the sum of their vertical components to zero.

Thus, if t represents the thickness of the material, r the internal radius of the shell, and w is the fluid pressure in pounds per square inch,

we have

$$\int_{\theta=0}^{\theta=\frac{\pi}{2}} (wxrd\theta) \cos\theta - p_h xt = 0,$$

whence

$$xtp_h = wxr,$$

and

$$\mathbf{p}_h = \frac{\mathbf{wr}}{\mathbf{t}}.$$

This is the **hoop tension** in pounds per square inch.

In this discussion the compression in the material of the shell, which is due to w and gradually diminishes from

w at the inner surface to zero at the outer surface, has been neglected (see page 176).

EXERCISE 192. Sketch as a free body a portion of the cylindrical shell whose dimensions are t, x, and $rd\theta$. From it deduce the hoop tension.

EXERCISE 193. What water pressure will a cast-iron pipe 36 inches in diameter and $1\frac{1}{16}$ inches thick withstand with a factor of safety of 15?

EXERCISE 194. A wrought-iron pipe 20 inches in diameter is to convey water under a head of 400 feet. If the factor of safety is to be 10, what should be its thickness?

EXERCISE 195. A boiler 72 inches in diameter, of steel whose ultimate tensile strength is 60,000 pounds per square inch, is to carry 150 pounds per square inch. The efficiency of the riveted joints is 80 % and the factor of safety should be 5. What should be the thickness of the plates?

EXERCISE 196. A cylinder to hold compressed air at 2000 pounds per square inch has a diameter of 7 inches, and the thickness of the metal is 0.35 inch. What is the working stress in the shell?

If the whole fluid pressure on the material closing the ends of a cylindrical shell is sustained by the shell (no stays passing from end to end parallel to the axis being introduced), a stress across planes perpendicular to the axis is produced in the shell. If the internal radius of the shell is r, the total tension due to an internal pressure of w pounds per square inch will be $w\pi r^2$ pounds. This force, if uniformly distributed over the area of material cut by a plane normal to the axis, which is nearly $2\pi rt$ square inches, will produce a **longitudinal tension,** p_l, in the shell. Hence

$$\mathbf{p}_l = \frac{\mathbf{wr}}{\mathbf{2\,t}} \text{ pounds per square inch.}$$

This longitudinal tension is one-half of the hoop tension. This accounts for the double riveting of longitudinal joints and the single riveting of circumferential joints in shells built up of plates.

When the material of the shell is in both longitudinal and hoop stress, then these are the principal stresses in the material.

According to theory I of elastic failure (see page 162), the dangerous stress would be the hoop stress p_h.

As there can exist no shear in this case (see page 147), theory II cannot be applied. Finally, according to theory III, the greatest strain produced in the material of the shell is $s_h' = \frac{p_h}{E} - \frac{1}{m}\frac{p_l}{E}$, and this determines the dangerous equivalent simple stress

$$p_\epsilon = p_h - \frac{1}{m} p_l = \left(2 - \frac{1}{m}\right)\frac{wr}{2t}.$$

If the material is wrought iron or steel when $\frac{1}{m} = .3$,

we have $$p_\epsilon = .85\frac{wr}{t}.$$

Exercise 197. Find the other equivalent simple stress for the above shell.

Exercise 198. A tank is 8 feet in diameter and 16 feet long. If the shell is 1 inch thick, find the hoop stress, the longitudinal stress, and the equivalent simple stress for an internal pressure of 160 pounds per square inch.

Exercise 199. The cast-iron air chamber of a pump is of cylindrical form with hemispherical ends. If the diameter is 10 inches and the length of the cylindrical part is 24 inches, what pressure can it withstand with a factor of safety of 5, the wall thickness being 0.5 inch?

Spherical Shells under Internal Pressure. — Let r be the internal radius of the shell, t the thickness of the wall, w the internal fluid pressure, p the stress across any plane passing through the center of the shell. Then

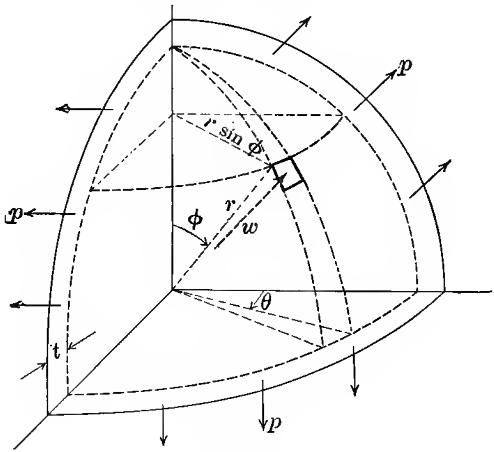


FIG. 89

from the equilibrium of the octant of the shell shown in Fig. 89 we have the sum of the vertical forces

$$\int_0^{\frac{\pi}{2}} wr^2 \sin\phi \cos\phi d\phi \int_0^{\frac{\pi}{2}} d\theta - \frac{p\,(2\,\pi rh)}{4} = 0.$$

For $\frac{2\,\pi rh}{4}$ is approximately the area over which the vertical stress p is assumed uniformly distributed, and the differential area upon which w acts is $(rd\phi)\,(r \sin\phi d\theta)$.

Thus

$$\mathbf{p} = \frac{\mathbf{wr}}{2\,\mathbf{t}}.$$

This, according to the first theory of elastic failure (page 162), is the dangerous stress, while according to the third theory the equivalent simple stress is

$$p_\epsilon = \left(1 - \frac{1}{m}\right)\frac{wr}{2t},$$

or as $\frac{1}{m} = .3$ for wrought iron and steel

$$p_\epsilon = .35\frac{wr}{t}.$$

EXERCISE 200. Sketch an element of the above shell as a free body and deduce the above value of p_ϵ.

EXERCISE 201. A force of 500 pounds applied to the plunger of a force pump is transferred to a hollow cast-iron sphere 10 inches in internal diameter. What should be the thickness of the shell if the plunger is 1 inch in diameter and the factor of safety is to be 6?

Cylindrical Shells under External Pressure. — The case of cylindrical shells under external pressure is analogous to the case of a bar in compression. If the shell is perfectly circular and of homogeneous material, its condition is similar to that of Euler's ideal column. As theoretical discussions of this case are complicated and lead to unsatisfactory results, they will not be attempted. Instead **empirical formulas** devised by R. T. Stewart (Trans. A. S. M. E., Vol. XXVII, 1906, page 730) will be quoted.

According to Stewart, for lap-welded Bessemer steel tubes

$$w = 1000\sqrt{1 - 1600\left(\frac{t}{d}\right)^2}. \qquad \text{(A)}$$

and

$$w = 86670\left(\frac{t}{d}\right) - 1386 \qquad \text{(B)}$$

where (A) is to be used when $w < 581$ or $\frac{t}{d} < .023$ and

(B) is to be used for values greater than these,

and w = the collapsing pressure in pounds per square inch,

d = the outside diameter of the tube in inches,

t = the thickness of the wall in inches.

EXERCISE 202. What should be the thickness of the wall of a 4-inch boiler tube in order that it may withstand a working pressure of 200 pounds per square inch with a factor of safety of 6?

EXERCISE 203. In a fire tube boiler the tubes are of steel 2 inches in external diameter and $\frac{1}{8}$ inch thick. Find the factor of safety when the pressure is 200 pounds per square inch.

Hollow Cylinders with Thick Walls, Lamé's Equations. — In cylindrical envelopes with thick walls the **hoop stress** cannot be assumed as uniformly distributed across planes passing through the axis of the cylindrical surfaces. Nor can the **radial stress** across cylindrical surfaces within the material, having the same axis as the boundary surfaces, be neglected, as in the above treatment of shells.

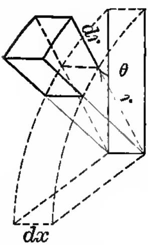


FIG. 90

Fig. 90 represents an element of the envelope, referred to cylindrical coördinates.

Let p_r denote the radial stress at a radius r, and p_h hoop stress at the same point of the material. Then if we neglect the longitudinal stress due to pressure on the material closing the ends of the envelope, there will be no stress across planes normal to the axis of the cylindrical surfaces, and Fig. 91 may be used to represent the element as a free body.

The sum of the radial forces acting upon this element must be zero, or

$$- p_r r\, d\theta\, dx + (p_r + dp_r)(r + dr)\, d\theta\, dx - p_h\, dr\, dx \sin\frac{d\theta}{2}$$
$$- (p_h + dp_h)\, dr\, dx \sin\frac{d\theta}{2} = 0.$$

As $\sin\frac{d\theta}{2} = \frac{d\theta}{2}$, it follows that

$$r\, dp_r + p_r\, dr - p_h\, dr = 0,$$

or
$$\frac{dp_r}{dr} + \frac{1}{r} p_r = \frac{1}{r} p_h. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This equation involves three variables, p_r, p_h, and r. Another equation is therefore required.

EXERCISE 204. What result is obtained by equating the sum of the tangential forces acting on the element, Fig. 91, to zero?

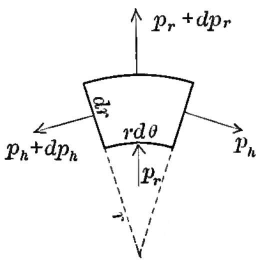


FIG. 91

To obtain another equation involving p_r and p_h, Lamé assumed that plane sections perpendicular to the axis of the envelope before the envelope was strained remained planes after distortion by the fluid pressure. This means that the actual strain parallel to the axis must be constant throughout the material. So that

$$-\frac{1}{m}\frac{p_r}{E} - \frac{1}{m}\frac{p_h}{E} = \text{a constant},$$

or
$$p_r + p_h = k, \text{ a constant}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Substituting the value of p_h from (2) into (1), we have

$$\frac{dp_r}{dr} + \frac{2}{r} p_r = \frac{1}{r} k.$$

Integrating this equation either by separation of the variables or as a linear differential equation, we obta n

$$p_r = \frac{k}{2} + \frac{c}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where c is the constant of integration.

By reason of (2) we also have

$$p_h = \frac{k}{2} - \frac{c}{r^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Assume now

w_i, as the internal fluid pressure,
w_ϵ, the external fluid pressure,
a, the external radius of the envelope,
and b, the internal radius of the envelope.

Then the constants k and c can be determined from equation (3) by noting that

when $r = a$, $p_r = - w_\epsilon$,
and when $r = b$, $p_r = - w_i$.

The negative signs indicate that the radial stresses are compressive at the outer and inner surfaces.

The above conditions substituted in (3) yield

$$- w_i = \frac{k}{2} + \frac{c}{b^2}$$

and

$$- w_\epsilon = \frac{k}{2} + \frac{c}{a^2}.$$

Solving these equations for k and c, we obtain

$$k = \frac{2(b^2 w_i - a^2 w_\epsilon)}{a^2 - b^2} \quad \text{and} \quad c = \frac{(w_\epsilon - w_i) a^2 b^2}{a^2 - b^2}.$$

So that, $$p_r = \frac{b^2 w_i - a^2 w_\epsilon}{a^2 - b^2} - \frac{a^2 b^2 (w_i - w_\epsilon)}{(a^2 - b^2) r^2},$$

$$p_h = \frac{b^2 w_i - a^2 w_\epsilon}{a^2 - b^2} + \frac{a^2 b^2 (w_i - w_\epsilon)}{(a^2 - b^2) r^2}.$$

These are Lamé's formulas for the radial and the hoop stresses in thick hollow cylinders under both external and internal fluid pressures.

Exercise 205. Find the hoop stress in the material of a hollow cylinder at 10, 13, 17, and 20 inches from the center. The external and internal radii of the envelope are 20 inches and 10 inches, while the only pressure is an internal fluid pressure of 2000 pounds per square inch.

Exercise 206. In Lamé's equation for the hoop stress, put $a = b + t$, where t, the thickness of the shell, is small compared to b, and show that $p_h = \frac{w_i (b + t)}{t}$ gives the value of the hoop stress at the dangerous surface when the external pressure is zero. How does this result compare with the result obtained on page 171?

Exercise 207. What is the approximate value of p_h at the outer surface of the shell described in Ex. 206?

Exercise 208. What are the values of the radial stresses at the outer and inner surfaces of a shell under internal pressure only as obtained from Lamé's equations?

Exercise 209. From Lamé's equations find a in terms of b, p_h, r, and w_i, assuming $w_\epsilon = 0$. From this result show that the internal fluid pressure can never exceed with safety the working stress of the material, provided elastic failure is assumed to be due to the greatest normal stress acting in the material.

Exercise 210. Find the greatest equivalent simple stress at the dangerous surface in a thick cylindrical envelope under internal pressure.

EXERCISE 211. Solve Ex. 194 by means of th formula developed in Ex. 210, and compare with the previous result.

EXERCISE 212. What pressure will a steel locomotive cylinder 22 inches in internal diameter and having walls $\frac{1}{4}$ inch thick withstand with a factor of safety of 10?

EXERCISE 213. Deduce from Lamé's equations the greatest equivalent simple stress at the dangerous point in a cylinder subjected to external pressure only.

EXERCISE 214. If the pressure on the material closing the ends of a cylindrical envelope is sustained by the envelope, show that the longitudinal tension is

$$p_l = \frac{w_i b^2 - w_\epsilon a^2}{a^2 - b^2}.$$

Under these conditions show that the equivalent simple hoop stress is

$$p_\epsilon = \frac{b^2 w_i - a^2 w_\epsilon}{3\,(a^2 - b^2)} + \frac{4\,a^2 b^2\,(w_i - w_\epsilon)}{3\,r^2\,(a^2 - b^2)},$$

when $m = 3$.

This formula is known as Clavarino's formula.

CHAPTER IX

THE PRINCIPLE OF WORK AS USED IN COMPUTING DEFLECTIONS

Section XX

DEFLECTIONS DUE TO BENDING

Whenever a body is strained work is done against the internal elastic forces (stresses) set up within the material. The energy stored in the material, provided the stresses do not exceed the elastic limit, is called the **resilience** of the material. Resilience is thus used to denote the **internal work** done upon the material in straining it. This is in distinction to the **external work,** the work performed by the external or applied forces acting upon the material and causing the strain.

Resilience of a Bent Beam, Neglecting Shear. — If we assumed the material of the beam to be under normal stresses only, the work done against these stresses in bending the beam can be computed as follows:

Consider a slice cut from the beam by two planes perpendicular to the axis and *dx* apart, as shown by the dotted lines in Fig. 92. Then after the beam is fully loaded the bending moments M_x and $M_x + dM_x$ will act upon the planes of section and the slice of the beam will be distorted to the shape shown by the solid lines in Fig. 92.

To avoid vibrations, the loading will be assumed to be gradually applied, so that the bending moments gradually increase from zero to the values M_x and $M_x + dM_x$ as the distortion of the slice progresses.

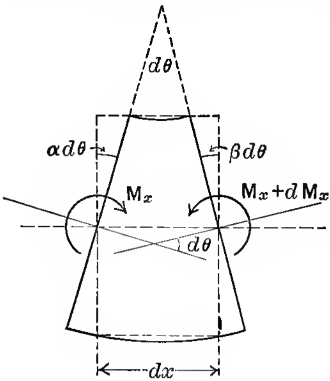


FIG. 92

Let $d\theta$ represent the total relative angular displacement of the planes of section; then $d\theta$ is also the change in the angle which the tangent to the elastic curve makes with the x-axis in passing from the points determined by x and $x + dx$. The plane sections may then be assumed to be displaced through the angles $\alpha d\theta$ and $\beta d\theta$ as shown in Fig. 92, where $\alpha + \beta = 1$.

As work done equals the product of the moment by the angular displacement, and as the average bending moments acting upon the element are $\frac{M_x}{2}$, $\frac{M_x + dM_x}{2}$, we have as the work done upon the elementary slice of the beam against the internal elastic forces

$$\frac{M_x}{2}(\alpha d\theta) + \left(\frac{M_x + dM_x}{2}\right)(\beta d\theta) = \frac{M_x}{2}(\alpha + \beta)\, d\theta = \frac{M_x}{2} d\theta.$$

But $d\theta$, the change in angle, is approximately equal to the change of slope of the elastic curve in the distance dx, or, expressed mathematically,

$$d\theta = d\left(\frac{dy}{dx}\right) = \left(\frac{d^2y}{dx^2}\right) dx = \left(\frac{M_x}{EI}\right) dx \text{ (see page 76)},$$

for we assume that the beam, originally straight, deflects but slightly.

Thus, the resilience of the whole beam is

$$\int \frac{1}{2} M_x \left(\frac{M_x}{EI}\right) dx = \frac{1}{2} \int \frac{M_x^2\, dx}{EI},$$

where the **integration must extend throughout the whole length of the beam.**

EXERCISE 215. Compute the resilience of a cantilever loaded only at the free end.

In computing the resilience of beams in which the integration must be carried on in several intervals, the selection of a new origin of coördinates for each interval often simplifies the algebraical work.

To illustrate, let us compute the resilience of a simple beam loaded only at mid-span.

In the first solution select O, Fig. 93, as origin for the

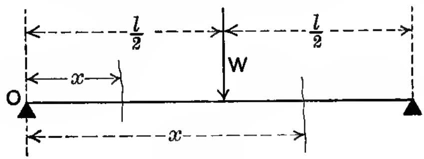


FIG. 93

work in both intervals. Then assuming the beam of constant cross section so that I is constant, we have

$$\text{Resilience} = \frac{1}{2\,EI}\int_0^{\frac{l}{2}} \left(\frac{Wx}{2}\right)^2 dx + \frac{1}{2\,EI}\int_{\frac{l}{2}}^{l} \left\{\frac{W(l-x)}{2}\right\}^2 dx$$

$$= \frac{W^2}{2\,EI}\left[\frac{l^3}{96} + \frac{l^3}{96}\right] = \frac{W^2 l^3}{96\,EI}.$$

In this, the second solution, consider the origin for the left-hand interval at A and for the right-hand interval at B, Fig. 94. Then the

$$\text{Resilience} = \frac{1}{2\,EI}\int_0^{\frac{l}{2}}\left(\frac{Wx}{2}\right)^2 dx + \frac{1}{2\,EI}\int_0^{\frac{l}{2}}\left(\frac{Wx}{2}\right)^2 dx$$

$$= \frac{1}{EI}\int_0^{\frac{l}{2}}\frac{W^2x^2}{4}dx = \frac{W^2l^3}{96\,EI}.$$

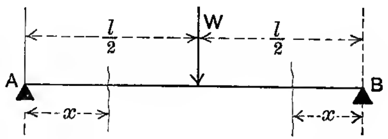


FIG. 94

EXERCISE 216. Compute the resilience of a uniformly loaded simple beam.

EXERCISE 217. Compute the resilience of a uniformly loaded simple beam bearing a concentrated load at mid-span.

EXERCISE 218. Why cannot the result of Ex. 217 be obtained by adding the resiliences due to each load separately?

Explain and compute the difference in these results.

Deflection under a Single Concentrated Load. — The gradual increase of the loading on a beam will cause a gradual increase in the strain (and in the stress) within the material of the beam. Under these conditions no degradation of the potential energy lost by the load as the beam deflects occurs; the whole of this energy is stored in the stressed material, always provided that the stresses within the material do not exceed its elastic limit.

Thus, the **external work** done by the external or applied forces **equals** the resilience of, or the **internal work** stored in, the material of the beam.

When a single concentrated load is gradually applied to a beam, the external work can readily be computed in terms of the load and the resulting deflection under the load. By equating this decrease in the potential energy of the load to the resilience due to the gradually applied load, the magnitude of the deflection can be found.

To illustrate, let us compute the deflection at the end of a cantilever loaded at the end only

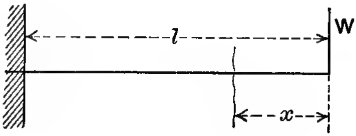


FIG. 95

From Fig. 95, assuming δ as the deflection under the load, we have the external work $= \dfrac{W\delta}{2}$

and the resilience $= \dfrac{1}{2\,EI}\displaystyle\int_0^l (Wx)^2\,dx$

$$= \frac{W^2l^3}{6\,EI}.$$

Whence $$\frac{W\delta}{2} = \frac{W^2l^3}{6\,EI}$$

and $$\delta = \frac{Wl^3}{3\,EI}.$$

EXERCISE 219. Find the deflection at mid-span of a simple beam loaded at mid-span.

EXERCISE 220. Find the deflection under the load for a simple beam, span l, loaded at a distance a from the left support with W pounds.

Deflection at Any Point under Any Loading. — From the preceding exercises it is evident that it is impossible to apply the method there used to beams loaded with more than one concentrated load, or even to finding the deflection at any point other than the one directly under the load, in a beam bearing only one concentrated load; for under these conditions the external work cannot be computed without involving unknown deflections other than the one sought.

To obtain a general formula for computing deflections by means of the principle of work, we may proceed as follows:

Consider a perfectly general loading, as shown in Fig. 96, and let the deflection δ under the load L be

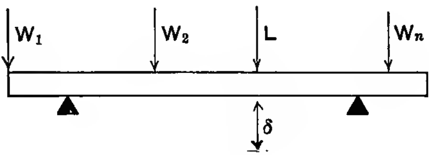


FIG. 96

required. For convenience of demonstration this L is singled out and specially designated, while all other loading will be referred to as W's.

It is then evident that the **external work due to L only,** all loads being gradually applied, will equal

$$\frac{L}{2}\delta,$$

where δ is the deflection under L due to all the loads including L and not the deflection due to L only.

If we can now compute the resilience or internal work due to L only, δ can be found.

Referring to Fig. 92, the internal work done upon an elementary slice of any beam is

$$\frac{1}{2}(M_x)(d\theta) = \frac{1}{2}(M_x)\left(\frac{M_x}{EI}dx\right),$$

or $\frac{1}{2}$ (bending moment at any section) (change in slope at that section).

If the total bending moment and the total change in slope are used, the internal work due to all loads is obtained. If, however, the bending moment, due to the load L only, and the total change in slope, due to all loads including L, are used, the result will be the internal work due to L only.

The last statement may be more readily understood from the following analysis:

Let ${}^{L}M_x$ represent the bending moment at any section due to the load L only and ${}^{n}M_x$ that due to W_n only, etc.

Consider the beam unloaded and the load L to be gradually applied, then the resilience per element due to this load is $\frac{1}{2}({}^{L}M_x)\left(\frac{{}^{L}M_x}{EI}dx\right)$. The load L remaining on the beam, apply W_1 gradually; then the **additional** resilience per element due to the additional displacement of the load L is $\frac{1}{2}({}^{L}M_x)\left(\frac{{}^{1}M_x}{EI}dx\right)$, for the bending moment due to L remains ${}^{L}M_x$ and the additional change in slope is due to the bending moment due to W_1, i.e., $\left(\frac{{}^{1}M_x}{EI}dx\right)$.

Thus the resilience per element due to the load L when both L and W_1 are on the beam is

$$\frac{1}{2}({}^{L}M_x)\left(\frac{{}^{L}M_x + {}^{1}M_x}{EI}dx\right),$$

and the resilience per element for any loading whatsoever due to L only must be

$$\frac{1}{2}({}^{L}M_x)\left(\frac{{}^{L}M_x + \sum_{m=1}^{m=n} {}^{m}M_x}{EI}dx\right),$$

as stated above.

Now the external work due to L only must equal the resilience due to L only, or

$$\frac{L\delta}{2} = \frac{1}{2}\int ({}^{L}M_x)\left(\frac{{}^{L}M_x + \sum_{m=1}^{m=n} {}^{m}M_x}{EI}dx\right).$$

But ${}^{L}M_x$, the bending moment at any section due to L only, is a function of x, so that we may write

$${}^{L}M_x = Lf(x).$$

EXERCISE 221. If L is applied to a simple beam, span l feet, at nl feet from the left abutment, find ${}^{L}M_x$ for any point in the span and indicate the value of $f(x)$.

So that,

$$\frac{L\delta}{2} = \frac{1}{2}\int Lf(x)\left\{\frac{Lf(x) + \sum_{m=1}^{m=n} {}^{m}M_x}{EI}\right\}dx,$$

and

$$\delta = \int \frac{Lf(x) + \sum_{m=1}^{m=n} {}^{m}M_x}{EI} f(x)\,dx, \quad . \quad . \quad (1)$$

where the integration must extend throughout the length of the beam.

To obtain $f(x)$, use its defining equation ${}^{L}M_x = Lf(x)$,

whence $$f(x) = \frac{{}^{L}M_x}{L}.$$

Note also (Ex. 221) that the value of $f(x)$ is independent of L, so that L may be assumed equal to unity as far as finding $f(x)$ is concerned. Thus,

$$f(x) = {}^{1}M_x.$$

That is, f (x) **is numerically equal to the bending moment at any section of the beam due to a hypothetical load of one pound placed at the section whose deflection is required.**

Return now to equation (1) for δ. This equation as it stands requires an actual load L on the beam at the point at which the deflection is sought. Such a load is evidently not always present. We have already shown that the $f(x)$ is independent of L; that is, it may be found by using a **hypothetical** load of one pound at the point whose deflection is sought. This hypothetical load should in no way be conceived as an actual load upon the beam producing an additional deflection. Its only purpose is the convenient determination of $f(x)$. Thus there is no reason why equation (1) should not hold for any value of L including the zero value. So that in order to obtain a general equation, giving the deflection at any point, loaded or unloaded, place L, in equation (1), equal to zero, whence

$$\boldsymbol{\delta = \int \frac{M_x f(x)}{EI} dx},$$

where M_x is the bending moment at any section of the beam due to **all** the loads upon the beam and **f (x) is numerically equal to the bending moment at any section due to a hypothetical one-pound load placed at the point at which the deflection** δ is sought, the integration to cover the whole length of the beam.

As an **application of this formula,** let us find the deflection at the right-hand end of the uniformly loaded beam illustrated in Fig. 97. Here the integration must be

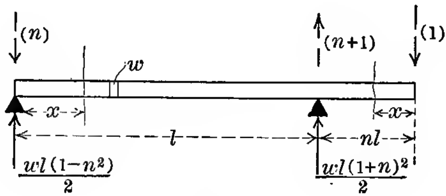


FIG. 97

performed in two stages, there being two intervals in which the values of both M_x and $f(x)$ are expressed by different functions of x.

Let us use the left end of the left interval as its origin and the right end of the right interval as its origin.

Great care must be taken not to confuse the effect of the hypothetical one-pound load with that of the actual loading on the beam. Find the reactions for each separately. These reactions are indicated in Fig. 97. All **hypothetical forces** are indicated by **dotted lines.**

Thus, in the two intervals we have

$$M_x = \frac{wl\,(1 - n^2)}{2}x - \frac{wx^2}{2}; \quad M_x = \frac{wx^2}{2},$$

and

$$f(x) = -\,nx; \qquad\qquad f(x) = x.$$

Note carefully the signs of the moment (see page 26). So that,

$$\delta = \frac{1}{EI}\int_0^l \left\{\frac{wl\,(1-n^2)}{2}x - \frac{wx^2}{2}\right\}\{-nx\}\,dx$$

$$+ \frac{1}{EI}\int_0^{nl}\left(\frac{wx^2}{2}\right)(x)\,dx$$

$$= \frac{1}{EI}\left[-\frac{wnl\,(1-n^2)\,x^3}{6} + \frac{wnx^4}{8}\right]_0^l + \frac{1}{EI}\left[\frac{wx^4}{8}\right]_0^{nl}$$

$$= \frac{wnl^4}{24\,EI}(3\,n^3 + 4\,n^2 - 1).$$

As **another application** let us find the equation of the elastic curve of a cantilever beam loaded at the free end only. This problem is solved by finding the deflection b at any point, at a distance a from the wall, Fig. 98.

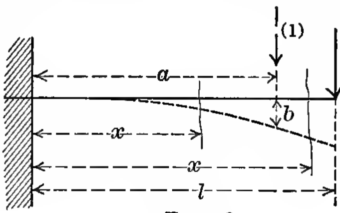


FIG. 98

Here, on account of the hypothetical one pound, two intervals must again be considered.

We have,

$$M_x = W\,(l - x); \qquad M_x = W\,(l - x),$$

also $$f\,(x) = (1)\,(a - x); \quad f\,(x) = 0.$$

So that,

$$b = \frac{1}{EI}\int_0^a W\,(l-x)\,(a-x)\,dx + \int_a^l W\,(l-x)\,(0)\,dx,$$

whence $$b = \frac{Wa^2}{6\,EI}(3\,l - a).$$

To put this in the usual form, put $a = x$ and $b = -y$, when

$$y = -\frac{Wx^2}{6EI}(3l - x).$$

This equation should be compared with the result obtained on page 78.

Exercise 222. A weightless cantilever is loaded at three-fourths of its length from the fixed end with W pounds. Find the deflection at the free end.

Exercise 223. Check the result of Ex. 222 by means of the formulas deduced on page 78.

Exercise 224. Find the maximum deflection of a uniformly loaded simple beam.

Exercise 225. Find the equation of the elastic curve for the left-hand interval of a simple beam loaded at mid-span with W pounds.

Exercise 226. A simple beam carries two loads of W pounds, one each at a inches from each support. The span is l inches. Find the deflection at mid-span.

Of course the general equation on page 189 may be used for beams of variable section, provided I can be expressed as a function of x.

Exercise 227. Find the greatest deflection of a cantilever of circular section, which tapers uniformly from a diameter d at the fixed end to $\frac{d}{2}$ at the free end; the only load considered to be W pounds at the free end.

SECTION XXI

DEFLECTIONS DUE TO SHEAR

The formulas giving the deflections of beams under various loadings are usually deduced from the equation $EI\frac{d^2y}{dx^2} = M_x$. This equation is deduced under the assumption that the beams are under the action of pure bending stresses only. Thus the deflections due to shearing stresses are not taken into account. The principle of work furnishes a simple means for the determination of these (up to this point disregarded) deflections.

Resilience Due to Shear. — In Fig. 99 is shown a differential element subject to a shearing stress, q. The **force** $q\ dy\ dz$ = **qdA** acting upon one face of this element suffers a **displacement** ϕ**dx,** if ϕ represents the shearing strain due to q. Thus the work done on, or the resilience of, this element due to shear is

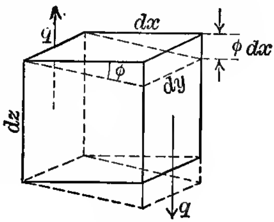


FIG. 99

$$\tfrac{1}{2}(qdA)(\phi\,dx),$$

provided we assume the loading and thus the shearing stress to increase gradually from zero to their greatest values.

As $G = \frac{q}{\phi}$, the resilience also equals $\frac{1}{2}\frac{q^2}{G}dx\ dA$, or, as it is often stated, the resilience per unit volume is $\frac{q^2}{2G}$.

EXERCISE 228. What is the resilience per unit volume for normal stress?

EXERCISE 229. Compute the resilience per unit volume for the element shown in Fig. 82.

Deflection Due to Shear. — As a **special case** consider the deflection at the free end of a cantilever (Fig. 100) due to shear only, the cantilever to be loaded at its free

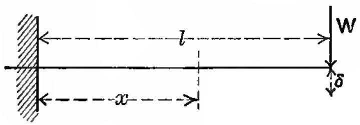


FIG. 100

end and to be regarded as weightless and of constant rectangular section.

If Fig. 99 represents an element of this beam, the internal work is readily seen to be

$$\int\int \frac{1}{2}(q dA)\left(\frac{q\,dx}{G}\right) = \int dx \int \frac{q^2 dA}{2G},$$

where the first integration must cover the section of the beam normal to its axis, and the second integration must be taken over the entire length of the beam.

The external work is $\frac{W\delta}{2}$, where δ is the required deflection. Thus,

$$\frac{W\delta}{2} = \int dx \int \frac{q^2 dA}{2G} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If we assume that the shearing force is uniformly distributed over the cross section, q is constant and equation (1) becomes

$$\frac{W\delta}{2} = \frac{q^2}{2G}\int dx \int dA,$$

where as $q = \frac{W}{A}$, under our assumption

$$\frac{W\delta}{2} = \frac{W^2}{2\,GA^2}\int_0^l dx\,(A),$$

or

$$\delta = \frac{Wl}{GA}.$$

This answer is only a very crude approximation to the true deflection due to shear, for on page 68 it was shown that the unit shearing stress is never constant over a beam section as above assumed.

For a rectangular section $b \times h$

$$q = \frac{W}{8\,I}(h^2 - 4\,y^2).$$

Using this value of q and placing $dA \doteq b\,dy$, equation (1) becomes

$$\frac{W\delta}{2} = \frac{W^2 b}{128\,I^2 G}\int_{x=0}^{x=l} dx \int_{y=-\frac{h}{2}}^{y=+\frac{h}{2}} (h^2 - 4\,y^2)^2\,dy.$$

or

$$\delta = \frac{Wb}{64\,I^2 G}\int_0^l dx\left[\frac{8}{15}\,h^5\right],$$

$$\delta = \frac{Wbh^5 l}{120\,I^2 G},$$

and as

$$I = \frac{h^3 b}{12},$$

$$\delta = \frac{6\,Wl}{5\,hbG} = \frac{6\,Wl}{5\,AG},$$

which is 20% greater than the value first found.

In **I-beams and built-up girders the whole shearing force** may be assumed **uniformly distributed over the sectional area of the web** (see page 70); the flanges are then assumed to carry no shearing stress. Under these conditions fairly correct results are obtained by means of the simpler calculations, neglecting the variation of the unit shearing stress.

EXERCISE 230. Compute the deflection due to shear at mid-span of a centrally loaded simple beam of rectangular section.

EXERCISE 231. Show that the total deflection at mid-span for the beam described in Ex. 230 is $\frac{Wl^3}{4\,Ebh^3}\left\{1 + 3\left(\frac{h}{l}\right)^2\right\}$ provided the material is such that $\frac{E}{G} = \frac{5}{2}$.

If $\left(\frac{h}{l}\right)$ is a small number, as is usually the case, how does this total deflection compare with the deflection due to bending only?

General Formula for the Deflection Due to Shear. — The method used above for finding the deflection due to shear can only be used when the beam carries a single concentrated load and the deflection sought occurs under this load.

Assume now a general loading, Fig. 96, and let us find the deflection under some load, say L. If δ is the total deflection due to shear under L, then the external work due to L and stored as shear resilience within the beam is $\frac{1}{2}\,L\delta$.

It is now necessary to compute the resilience due to the sinking of the load L only. Consider any element of the beam, as in Fig. 99, then its resilience due to any shearing stress q is

$$\frac{1}{2}\,(\text{force})\,(\text{displacement}) = \frac{1}{2}\,(q\,dA)\left(\frac{q}{G}\,dx\right).$$

Let ${}^{L}q$ be the shearing stress due to the load L,

${}^{1}q$ be the shearing stress due to the load W_1, etc.;

then the resilience per differential element, provided L is the only load upon the beam, is

$$\frac{1}{2}({}^{L}q\,dA)\left(\frac{{}^{L}q}{G}\,dx\right).$$

When the load W_1 has been gradually but wholly applied, the additional resilience per element due to the additional sinking of the load L only, will be

$$\frac{1}{2}({}^{L}q\,dA)\left(\frac{{}^{1}q}{G}\,dx\right),$$

where $\left(\frac{{}^{1}q}{G}\,dx\right)$ is the additional displacement of the shearing force, ${}^{L}q\,dA$, this force being due to the load L only.

The resilience due to the load L displaced by δ under action of all loads upon the beam will then be

$$\frac{1}{2}({}^{L}q\,dA)\left(\frac{{}^{L}q + \sum\limits_{m=1}^{m=n} {}^{m}q}{G}\,dx\right) \text{ per element.}$$

In general the vertical component of the shearing stress at any point within a beam may be expressed in the form

$$q = \frac{Q_x}{bI}\int y\,dA \qquad \text{(see page 66).}$$

Substituting this value of q, the resilience per element due to L becomes

$$\frac{1}{2G}\,\frac{{}^{L}Q_x}{{}^{L}Q_x + \sum\limits_{m=1}^{m=n} {}^{m}Q_x}\left\{\frac{{}^{L}Q_x + \sum\limits_{m=1}^{m=n} {}^{m}Q_x}{bI}\int y\,dA\right\}^2 dA\,dx.$$

Now ${}^{L}Q_x$ being the shearing force on any section due to the load L only, we may put

$$ {}^{L}Q_x = L\phi(x), $$

and thus

$$ \delta = \int \frac{\phi(x)}{\left({}^{L}Q_x + \sum_{m=1}^{m=n} {}^{m}Q_x\right)G} dx $$

$$ \int \left\{ \frac{{}^{L}Q_x + \sum_{m=1}^{m=n} {}^{m}Q_x}{bI} \int y\,dA \right\}^2 dA. $$

It may happen that no load L exists at the point at which the deflection is desired. Under these conditions placing $L = 0$ in the above equation simply removes the term ${}^{L}Q_x$.

Moreover, as ${}^{L}Q_x + \sum_{m=1}^{m=n} {}^{m}Q_x$ always represents the total shearing force at any section due to all existing loads, we may replace it by the symbol Q_x previously used.

The formula for the deflection can then be written

$$ \delta = \int \frac{\phi(x)}{Q_x G} dx \int \left(\frac{Q_x}{bI} \int y\,dA \right)^2 dA. $$

Here the expression within the parenthesis must first be evaluated as on page 68, and expressed in terms of y. Then, when dA is expressed in terms of y, the integration indicated may be extended over the beam section. The result of this integration is either a constant or it must be expressed in terms of x when the beam section varies. This allows the final integration to be extended over the whole length of the beam.

The $\phi(x)$ used in the above formula is defined by ${}^{L}Q_x = L\phi(x)$, or when L is unity

$$\phi(x) = {}^{1}Q_x;$$

that is, ϕ (**x**) **is the shearing force at any section of the beam due to a hypothetical load of one pound placed at the point at which the deflection is to be computed.**

In case the above formula is to be used **when the variation of the shearing stress over the beam section is** so small as to be **negligible** (as in the case of **I**-beams, etc.), $\frac{Q_x}{bI}\int y\,dA$ may be put equal to $q = \frac{Q_x}{A}$, a constant, provided the area subject to shear is constant, and then

$$\delta = \int \frac{Q_x\phi(x)}{AG}\,dx,$$

where A is that portion of the beam section assumed to carry the whole uniformly distributed shear.

As an **application** of the above formula, let us find the deflection at mid-span of a simple uniformly loaded beam, Fig. 101.

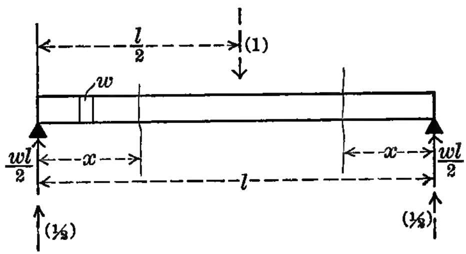


FIG. 101

As the deflection at mid-span is required, the hypothetical load of one pound must be there applied. This load and its reactions are indicated by means of dotted lines; care must be taken to remember that these forces are not actual loads upon the beam, and in no wise add to the deflection about to be computed.

In the two intervals to be considered we have

$$Q_x = \frac{wl}{2} - wx, \qquad Q_x = -\left(\frac{wl}{2} - wx\right),$$

and $\quad \phi(x) = \frac{1}{2}, \qquad \phi(x) = -\left(\frac{1}{2}\right).$

If the **variation of the shearing stress over the beam section is neglected** and A is the constant sectional area subject to shear, we have the required deflection

$$\delta = \int \frac{Q_x \phi(x)}{AG}\, dx = \frac{1}{AG}\int_0^{\frac{l}{2}} \left(\frac{wl}{2} - wx\right)\left(\frac{1}{2}\right) dx$$

$$+ \frac{1}{AG}\int_0^{\frac{l}{2}} - \left(\frac{wl}{2} - wx\right)\left(-\frac{1}{2}\right) dx$$

$$= \frac{1}{AG}\int_0^{\frac{l}{2}} \left(\frac{wl}{2} - wx\right) dx = \frac{1}{AG}\left(\frac{wl^2}{8}\right) = \frac{wl^2}{8\,AG}.$$

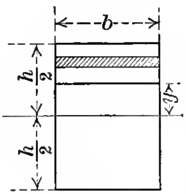


FIG. 102

If a more accurate solution is desired, the shape of the section must be considered and $\int y\, dA$ must first be evaluated.

For a beam of **constant rectangular section** b inches wide and h inches deep, Fig. 102,

$$\int y\, dA = \int_y^{\frac{h}{2}} yb\, dy = \frac{b}{8}\left(h^2 - 4\,y^2\right),$$

and $$\delta = \int \frac{\phi(x)}{Q_x G}\, dx \int \left(\frac{Q_x}{bI}\int y\, dA\right)^2 dA,$$

$$= \int \frac{\phi(x)}{Q_x G} dx \int_{-\frac{h}{2}}^{+\frac{h}{2}} \frac{Q_x^2}{64 I^2} (h^2 - 4y^2)^2 b\, dy,$$

$$= \int \frac{bQ_x \phi(x)}{I^2 G} dx \left[\frac{6}{5} h^5\right],$$

and as $$I = \frac{h^3 b}{12},$$

$$\delta = \frac{6}{5} \int \frac{Q_x \phi(x)}{AG} dx.$$

This is the **general form of the equation for** a beam of **constant rectangular section.**

Substituting the values of Q_x and $\phi(x)$ above derived from Fig. 101 and integrating between proper limits, we obtain

$$\delta = \frac{3}{20} \frac{wl^2}{AG}.$$

EXERCISE 232. Show that the general equation for the deflection due to shear in beams of *constant circular section* is

$$\frac{10}{9} \int \frac{Q_x \phi(x)}{AG} dx.$$

EXERCISE 233. What is the deflection due to shear at the free end of a uniformly loaded cantilever beam, neglecting the variation of shearing stress over the cross section?

EXERCISE 234. A uniformly loaded beam overhangs one abutment by a inches. The distance between the abutments is l inches. Find the deflection due to shear at the free end.

PROBLEMS FOR REVIEW

235. A hollow cast-iron column (so short that buckling need not be feared) must sustain a steady load of 50,000 pounds and is to rest upon a sandstone base. Assuming the working stress of sandstone as 60 pounds per square inch and the bearing power of the dry clay soil supporting the foundation as 3 tons per square foot, find the sectional area of the column, the bearing surface of the column footing, and that of the sandstone base.

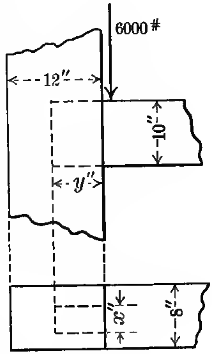


FIG. 103

236. Find the dimensions x and y indicated in Fig. 103. The material is wood; the load is variable.

237. Fig. 104 shows the lower end of a foundation bolt subject to a constant tension. Write the equations whose solution will yield the dimensions d, m, n, x, y, and z, in terms of the tension T pounds, assuming the bolt and cotter to be wrought iron and anchored to a steel plate.

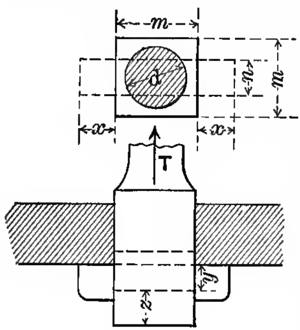


FIG. 104

238. Deduce expressions for the change in sectional area and the change in volume produced in a cylinder (radius, r; length, l) by a longitudinal compressive stress of p pounds per square inch. The constants of the material are E and $\frac{1}{m}$.

239. A wrought-iron bar 18 feet long and 1.5 inches in diameter is heated to 200° F. and its ends are then firmly fixed. The coefficient of expansion is .0000068. What force will this bar exert when it has cooled to 100° F.?

240. The compression flange of a cast-iron beam is 4 inches wide and 2 inches deep; the tension flange 12 inches wide and 3 inches deep; the web 10 inches by 2 inches. (a) Locate the neutral axis of the section. (b) Find the second moment of area about the neutral axis. (c) What load per foot run can this beam carry on a 10-foot span, the greatest tensile stress not to exceed 2000 pounds per square inch? (d) What is the greatest compressive stress in the beam under the loading computed in (c)?

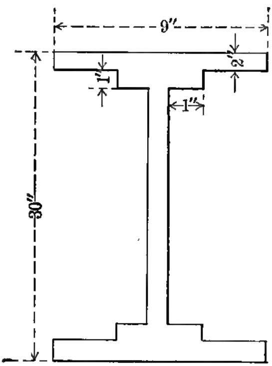


FIG. 105

241. The shearing force to be resisted by the section shown in Fig. 105 is 100,000 pounds. Compute the unit shearing stress at (a) 2.+ inches, (b) 3.+ inches, (c) 15 inches from the top of the section. What is the mean shear in the web if the web is assumed to carry the whole shear?

242. Compute the deflection at A and at B for the loading shown in Fig. 106.

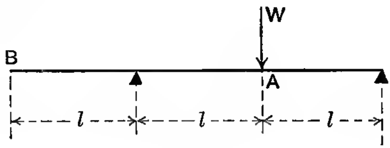


FIG. 106

243. A shaft AB rests on two supports C and D, Fig. 107, and is loaded at the ends as shown. How much higher is the

middle of the shaft than the ends? Solve this problem by means of formulas deduced on pages 80 to 84.

244. A short post 12 inches square carries 28,800 pounds. The center of pressure is 3 inches from one edge and 6 inches from another edge of the top section. Find the greatest, the least, and the mean unit stress.

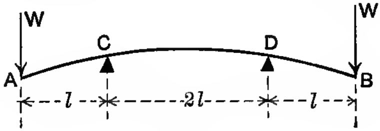


FIG. 107

245. A cantilever carrying a uniformly distributed load is propped to the level of the fixed end at a point $\frac{3}{4}$ of its length from the fixed end. What fraction of the whole load does this prop carry?

246. Sketch the shearing force and bending moment diagrams for the propped beam described in Ex. 245, and compute the principal values of the shearing forces and bending moments.

247. The turbine shaft of a 5-horse-power De Laval steam turbine makes 30,000 r.p.m. What should be its diameter if the working strength of steel in shear is 3000 pounds per square inch?

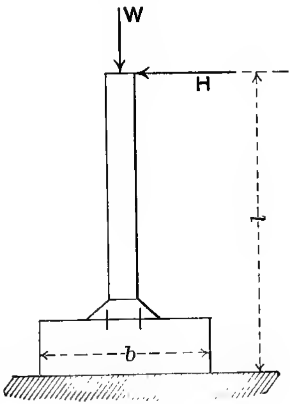


FIG. 108

248. Fig. 108 represents a column and its foundation. Find the dimension b and the relation which must exist between W, H, l, and p_e, if p_e is the greatest allowable pressure on the soil in pounds per unit area. The pressure between

the foundation and the soil may be assumed to vary according to the straight-line law.

249. A column is built up of 4 **Z**-bars and 3 plates as shown in Fig. 109. The area of the **Z**-bar section is 6.96 square inches, and its moments of inertia about the indicated axes are $I_1 = 23.68$ inches^4 and $I_2 = 11.37$ inches^4. The plates are $7\frac{3}{8}$ inches by 1 inch and 13 inches by 1 inch. Find the least radius of gyration of this column section and the load it can safely support on an unsupported length of 20 feet, with a working stress of 16,000 pounds per square inch, the ends being fixed. Use Rankine's formula.

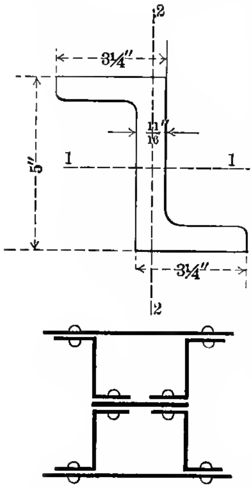


FIG. 109

250. A locomotive driving wheel 60 inches in diameter has a steel tire, $\frac{3}{4}$ inch thick, shrunk upon it. The diameter of the tire was originally $\frac{1499}{1500}$ of the diameter of the wheel. Find the hoop stress produced in the tire and the pressure in pounds per square inch on the wheel.

251. A rod of iron 1 inch in diameter and 6 feet long is found to stretch $\frac{1}{16}$ of an inch under a load of 7.5 tons. The ultimate strength of this iron is 50,000 pounds per square inch. Find its modulus of elasticity.

252. A beam 20 feet long bears a uniform load of 100 pounds per foot on the left half of its length. Find the bending moments at 5, 10, and 15 feet from its left end. What is the maximum bending moment and where does it occur?

253. Find the necessary thickness of a copper steam pipe 4 inches in diameter for a steam pressure of 100 pounds per

square inch. The safe stress for copper may be taken at 1000 pounds per square inch.

254. Find the greatest shearing stress in a circular shaft 1 inch in diameter when transmitting 1 horse power at 100 r.p.m.

255. What is the deflection at the middle of a 2 by 12 inch pine joist of 12-foot span supported at the ends and uniformly loaded with 3200 pounds?

256. A cylinder 9.5 inches inside diameter contains steam at 180 pounds per square inch. The cylinder head is held by 6 wrought-iron bolts placed at equal distances from each other on the flange. Find the diameter and the depth of the head of the bolts for a factor of safety of 10 against shear and tension.

257. Sketch the shearing force and bending moment diagrams and compute the greatest bending moment and the location of the dangerous section for a simple beam, span 20 feet, loaded with 100 pounds per foot for a length of 8 feet starting at one abutment.

258. A floor designed to support a total load of 200 pounds per square foot is to be supported by steel **I**-beams having a span of 12 feet and spaced 5 feet center to center. What should be the section modulus of these beams with a factor of safety of 5?

259. A hollow steel shaft (length 5 feet, outside diameter 6 inches) is to transmit 300 horse power at 200 r.p.m. What should be its internal diameter, allowing a factor of safety of 6? Through what angle (expressed in degrees) will this shaft be twisted while transmitting this power?

260. Show that a prism under compression is also in shear.

261. A simple wooden beam 10 feet long and 6 inches deep is to carry a load of 700 pounds at mid-span with a factor of safety of 8. How wide must it be? If the weight of the beam is 40 pounds per cubic foot and this weight is included in the computation, what is its width?

262. Find the dangerous section and the greatest bending moment for the loading shown in Fig. 110; sketch the shearing force and bending moment diagrams.

263. A flanged cylinder 10 inches inside diameter and 10 feet long contains steam at 150 pounds per square inch. The heads of the cylinder are held against the flanges by a single wrought-iron bolt, 1 inch in diameter, passing

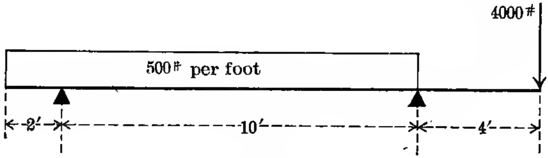


FIG. 110

through the cylinder. Assuming the heads to be rigid, how much must the bolt be stretched by screwing up the nuts in order that the heads may be held steam-tight against the flanges?

264. What horse power can be transmitted, with a factor of safety of 6, by a wrought-iron shaft 4 inches in diameter when making 110 r.p.m.?

265. Two 8-inch steel I-beams, 25.25 pounds per foot, area of section 7.43 inches^2, moments of inertia 68 inches^4 and 4.71 inches^4, are joined by lattice work to form a column 20 feet long. How far apart must these beams be placed, center to center, in order that the column may resist buckling in one direction as well as in another?

266. The total load on the axle of a truck is 6 tons; the wheels are 6 feet apart and the axle boxes 5 feet apart. Draw the bending moment and shearing force diagrams and compute the bending moment midway between the wheels.

267. A cast-iron pipe 18 inches in diameter is to be used to transmit water under a head of 300 feet. If the factor of safety is to be 15 and the ultimate strength of cast-iron

in tension is 20,000 pounds per square inch, what should be the thickness of the metal?

268. What deflection at mid-span due to a uniformly distributed load would produce a stress of 800 pounds per square inch in the extreme fibers of a 2 by 10 inch simple wooden beam of 20-foot span? (E = 1,700,000.)

269. Compute the distance of the dangerous section from the left support of a simple beam, span l feet, uniformly loaded for a distance of a feet starting at b feet from the left support.

270. Compute, by means of the principle of work, the deflection at mid-span of a uniformly loaded simple beam, the deflection at mid-span of a centrally loaded simple beam. From these results compute the reactions of the supports of a beam continuous over two equal spans of l feet and uniformly loaded with w pounds per foot run.

271. Assuming that a chain is twice as strong as the round bar of which the links are made and that the working strength of the metal is 6000 pounds per square inch, what should be the diameter of the metal in the chain used on a 20-ton crane with three-sheaved blocks?

272. Compute the position of the horizontal neutral axis, the second moment of area about this axis, and the section modulus of the section illustrated in Fig. 111.

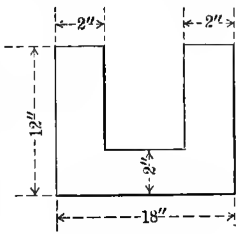


FIG. 111

273. What must be the section modulus of a mild steel beam designed to carry concentrated loads of 20 tons at 5 feet from the abutments of a 30-foot span, the fiber stress not to exceed 16,250 pounds per square inch?

274. The diameter of a solid steel shaft designed to transmit 9000 horse power at 140 r.p.m. with a working stress of 10,000 pounds per square inch is 14.6 inches. The greatest

twisting moment is assumed to be $\frac{3}{2}$ of the mean. If this shaft is replaced by a hollow one whose internal diameter is $\frac{9}{16}$ of the external and the shafts are 60 feet long, what weight is saved? (Steel weighs 480 pounds per cubic foot.)

275. Find, by means of the principle of work, the deflection at the free end of a cantilever loaded at the free end, the deflection of a cantilever uniformly loaded, and from these results compute the pressure on a prop which sustains the free end of a uniformly loaded cantilever at the level of the fixed end.

276. Compute by means of Rankine's formula the load a 10-foot wrought-iron pipe 4 inches external and 3 inches internal diameter can sustain when used as a column.

ANSWERS

1. $ML^{-1}T^{-2}$; $M^{\circ}L^{\circ}T^{\circ}$; $ML^{-1}T^{-2}$.
2. 16,530,000 lbs. per sq. in.
3. 25,000 lbs.
4. 69 ft.
5. .2 in.
7. .0965 in.; .00142 sq. in.; .158 cu. in.
8. 24,800 lbs.
9. 230.5 ft.
10. 2.16 in.
11. 5 in. by 5 in.; 12 in. by 12 in.; under variable loads.
12. 8 in.
13. 47,200 lbs.
14. .0026 in.
15. .25 in.
16. 14,980 lbs.
17. 154° F.
18. 58,500 lbs. per sq. in.
20. 109 ft.-lbs.
21. 1.53 H.P.
24. $\frac{pl}{E}$.
26. 250 sq. in.; 270 sq. in.
27. Practically 800 lbs. per sq. in.
28. $p = \frac{Wl \pm \sqrt{W^2l^2 + 2\,AWhlE}}{Al}$.
34. $\sqrt{\left(\frac{l_2}{2}\right)^2 - l_1^2}$, from the center.
37. $Q_x = -\frac{wx^2}{l} + \frac{wl}{4}$; $M_x = -\frac{wx^3}{3\,l} + \frac{wlx}{4}$; $M_{\max} = \frac{wl^2}{12}$.

39. 6200 lbs.-ft.; 2750 lbs.-ft.

41. At $\frac{3\,l}{8}$; $M_{\max} = \frac{9}{128} wl^2$.

42. 18.33 ft.; 5549 lbs.-ft.

43. 4000 lbs.-ft.; 192,000 lbs.-ft.; at right-hand abutment.

46. $8\frac{1}{3}$ in. (1 inch = 10,000,000 lbs.-in.)

47. 1.84 in., .84 in.

48. 1.49 in. from outer surface of the plate.

52. 638 in.^{4}, 53.4 in.^{4}

53. 4.95 in. from top; 373.5 in.^{4}

54. 7 in.

55. 780 lbs.

56. 5.7 in.

57. 825 lbs.

58. 1590 lbs.

59. 3.28; 1.92.

60. $bh^2 = 157$ ∴ 2 in. × 10 in.

61. 3.38 in., say 4 in.

62. $bh^2 = 972$ ∴ either 3 in. × 18 in. or 4 in. × 16 in.

63. 30.75 in.^{3}

64. 8820 in.^{3}

65. 6025 lbs.

66. $d\sqrt{\frac{2}{3}}$; $d\sqrt{\frac{1}{3}}$.

68. 8 ft. from the ends.

70. $h = x\sqrt{\frac{3\,w}{bp_c}}$, or $b = \frac{3\,wx^2}{h^2 p_c}$.

71. $\frac{x^2}{\left(\frac{l}{2}\right)^2} + \frac{y^2}{\frac{3\,wl^2}{4\,bp_c}} = 1$, $\sqrt{\frac{3\,wl^2}{4\,bp_c}}$.

72. 7200 lbs. per sq. in.

73. 70.3, 93.7, 0, lbs. per sq. in.

75. 104, 156 lbs. per sq. in.

76. 8, 2.7.

77. 2.5 in., 6.7.
78. 3 in., 11.7.
79. 3.6, 12, 16.2, 48.6, 49.2, 0, each multiplied by $\frac{Q_x}{I}$.
80. 2.92, .65.
81. $\frac{l}{h}$.
82. $\frac{4l}{h}$.
83. $\frac{2l}{h}$.
84. $y = -\frac{W}{6EI}(x^3 - 3l^2x + 2l^3)$.
85. .00604, .241 in.
86. .257 in.
87. $y = \frac{w}{24EI}(-x^4 + 4l^3x - 3l^4)$, origin at free end.

$y = \frac{w}{24EI}(-x^4 + 4lx^3 - 6l^2x^2)$, origin at fixed end.

$-\frac{wl^3}{6EI}, \quad -\frac{wl^4}{8EI}, \quad -\frac{p_c l^2}{4Ec}$.

88. $y = \frac{wx}{24EI}(-x^3 + 2lx^2 - l^3)$.

$\pm\frac{wl^3}{24EI}, \quad -\frac{5wl^4}{384EI}, \quad -\frac{5p_cl^2}{48Ec}$.

89. 6.94 ft., 3.9 = factor of safety.
90. 20.8 ft., 12,500 lbs.
91. $y = \frac{W}{48EI}(l^3 - 9l^2x + 12lx^2 - 4x^3)$, for right-hand interval.

$y = \frac{W}{48EI}(4x^3 - 3l^2x)$, for left-hand interval.

$-\frac{Wl^2}{16EI}, \quad -\frac{Wl^3}{48EI}, \quad -\frac{p_cl}{12Ec}$.

92. $-\frac{W(l-a)\{l^2-(l-a)^2\}}{9lEI}\sqrt{\frac{l^2-(l-a)^2}{3}}$.

93. $-\frac{3\,Wl^3}{256\,EI}$, $-\frac{7\,Wl^3}{768\,EI}$, $-\frac{5\sqrt{5}\,Wl^3}{768\,EI}$.

95. $-\frac{Wl^3}{48\,EI}$.

96. $-\frac{Wa}{24\,EI}(3\,l^2 - 4\,a^2)$.

97. $\left(\frac{5}{384} + \frac{1}{128}\right)\frac{Wl^3}{EI} = \frac{Wl^3}{48\,EI}$.

98. $2.12 + 1.59 = 3.7$ in.

99. $\frac{5\,W}{16}$.

100. $\frac{3}{8}wl$, $\frac{5}{4}wl$, $\frac{3}{8}wl$.

101. $\frac{5}{16}W$, $\frac{11}{8}W$, $\frac{5}{16}W$.

102. $\frac{351}{2048}wl$.

104. $\frac{5}{16}W$; $\frac{5}{32}Wl$, $-\frac{6}{32}Wl$; $.448\,l$, $.00933\,\frac{Wl^3}{EI}$; $\frac{8}{11}l$.

105. $\frac{wl}{2}$; $\frac{wl^2}{12}$; $\frac{wl^2}{24}$; $-\frac{wl^4}{384\,EI}$; $\frac{l}{2} \pm \frac{l}{2}\sqrt{\frac{1}{3}}$, $.211\,l$ from the ends.

$-\frac{p_c l^2}{32\,Ec}$.

107. $\frac{11}{28}wl$, $\frac{8}{7}wl$, $\frac{13}{14}wl$, $\frac{8}{7}wl$, $\frac{11}{28}wl$.

108. $-\frac{wl^3}{4\,l_1(3l + 2\,l_1)}$.

113. $\frac{k^2}{c}$, $\frac{ch_2}{k^2} > 1$.

114. $\frac{1}{1 + \frac{ch_1}{k^2}}$.

115. $c = \frac{h}{6}$.

116. $c = \dfrac{R^2 + r^2}{4R}$.

117. 3230 and 1410 lbs. per sq. in.

118. 14,834 comp., 13,166 tension.

119. $h^2 = s(t_2^2 + t_1t_2 - t_1^2)$.

120. 2 in.

121. 11.7 in.

125. 95,600 lbs.

126. 5.58 in.

127. 296, 148.

128. $k^2 = 2.21$, $\dfrac{l}{k} = 145$, 5430 lbs. per sq. in.

129. 1300 lbs., 9600 lbs.

130. $\dfrac{l}{k} = 57$, 17.3 in.

131. By Euler's formula, 3.12 in.; $\dfrac{l}{k} = 77$; by Rankine's formula 3.5 in.

132. $\dfrac{l}{k} = 75$; 227,000 lbs.

134. $\dfrac{d_1^2 + d_2^2}{d_1\sqrt{d_1^2 - d_2^2}}$.

135. 1.44.

136. 9800 lbs. per sq. in.; 13° 46′.

137. $\dfrac{d_1^2 + d_2^2}{d_1^2 - d_2^2}$.

138. 1.67.

140. $2\frac{1}{16}$ in.

142. 1.68 H.P.

143. 5.4 in.

144. 8.4.

145. 12,000,000 lbs. per sq. in.

146. 2.37°.

147. 4.1 in.; 12.5.

151. $p_x = -279$; $q = -16.3$; $p' = -29.5$; $q' = -87.5$.

153. $\alpha = 3^\circ 20'$, $p_1 = +.9$; $\alpha = 93^\circ 20'$, $p_2 = -280.9$
159. $\alpha = 48^\circ 20'$, -140.9; $\alpha = 138^\circ 20'$, $+140.9$.
164. 4.3; 5.6.
165. $p' = p$; $q' = 0$.
167. 3600 lbs. per sq. in.
168. 2; 1.3.
169. $\mu = \cot 2\alpha$.
170. .36.
171. 147,000 lbs. per sq. in.
172. 65,300 lbs. per sq. in.
177. $s_1' = s_1 - \frac{s_2}{m} - \frac{s_3}{m}$; $s_2' = s_2 - \frac{s_1}{m} - \frac{s_3}{m}$; $s_3' = s_3 - \frac{s_1}{m} - \frac{s_2}{m}$.
180. .304.
181. .2.
185. 8750 lbs. per sq. in.
186. $p_1 + \frac{1}{m}(-p_2 + p_3)$; $p_2 + \frac{1}{m}(-p_1 + p_3)$;

$$-p_3 - \frac{1}{m}(p_1 + p_2).$$

191. 9780; 17,900; 22,300, − 4250; 26,500; 23,500, − 11,000.
193. 88.6 lbs. per sq. in.
194. .32 in.
195. $\frac{9}{16}$ in.
196. 20,000 lbs. per sq. in.
197. $\left(1 - \frac{2}{m}\right)\frac{wr}{2t}$, $.2\frac{wr}{t}$.
198. 6530 lbs. per sq. in.
199. 405 lbs. per sq. in.
201. .339 in., or .423 in.
202. .119 in.
203. 20.
204. $p_h = c$.
205. $p_h = 666 + \frac{266000}{r^2}$; 3330, 2240, 1590, 1330.
207. $\frac{w_i b}{t}$.

208. $0, -w_i$.

210. $p_\epsilon = \dfrac{w_i}{a^2 - b^2}\left\{\left(1 + \dfrac{1}{m}\right)a^2 + \left(1 - \dfrac{1}{m}\right)b^2\right\}$.

211. .31 in.

212. 143 lbs. per sq. in.

213. Outside $p_\epsilon = -\dfrac{w_\epsilon}{a^2 - b^2}\left\{\left(\dfrac{1}{m} - 1\right)a^2 + \left(\dfrac{1}{m} + 1\right)b^2\right\}$,

inside $p_\epsilon = -\dfrac{2\,w_\epsilon a^2}{a^2 - b^2}$.

215. $\dfrac{W^2 l^2}{6\,EI}$.

216. $\dfrac{w^2 l^5}{240\,EI}$.

217. $\dfrac{1}{EI}\left\{\dfrac{W^2 l^3}{96} + \dfrac{5\,Wwl^4}{384} + \dfrac{w^2 l^5}{240}\right\}$.

219. $\dfrac{Wl^3}{48\,EI}$.

220. $\dfrac{Wa^2(l - a)^2}{3\,lEI}$.

221. $L\{(1 - n)\,x\}$; $L\{n\,(l - x)\}$.

222. $\dfrac{27}{128}\dfrac{Wl^3}{EI}$.

224. $\dfrac{5}{384}\dfrac{wl^4}{EI}$.

225. $y = \dfrac{W}{48\,EI}(4\,x^3 - 3\,l^2 x)$.

226. $\dfrac{Wa\,(3\,l^2 - 4\,a^2)}{24\,EI}$.

227. $\dfrac{128\,Wl^3}{3\,\pi d^4 E}$.

228. $\dfrac{p^2}{2\,E}$.

230. $\dfrac{3}{10}\dfrac{Wl}{GA}$.

233. $\dfrac{wl^2}{2\,GA}$.

234. $\dfrac{Wa^2\,(a+l)}{2\,lGA}$.

235. 4.2 sq. in.; 834 sq. in.; 1200 sq. in.

236. 5 in.; 2 in.

237. $T = 13{,}500\,\pi d^2 = 13{,}500\,(m^2 - mn) = 10{,}000\,(2\,qn)$.
$= 10{,}000\,(2\,zm) = 10{,}000\,(mn) = 10{,}000\,(2\,xn)$.

238. $\dfrac{2\,p\pi r^2}{mE}$; $\dfrac{p(m-2)\,\pi r^2 l}{mE}$.

239. 33,600.

240. 5.09 in. from bottom; 1465.9 in.^{4}; 3840 lbs. per foot run; 3890 lbs. per sq. in.

241. 916, 3160, 3940, 4170 lbs. per sq. in.

242. $\dfrac{Wl^3}{6\,EI}$, $\dfrac{Wl^3}{4\,EI}$.

243. $\dfrac{11}{6}\,\dfrac{wl^3}{EI}$.

244. 500, − 100, 200 lbs. per sq. in.

245. $\dfrac{19}{32}$.

247. .262 in.

248. $b = \dfrac{6\,Hl}{W}$, $W^3 \equiv 18\,p\epsilon H^2 l^2$.

249. 3.51 in., 847,000 lbs.

250. 20,000, 500 lbs. per sq. in.

251. 22,300,000 lbs. per sq. in.

252. 2500, 2500, 1250, lbs.-ft.; 7.5 ft., 2812.5 lbs. per sq. in.

253. .2 in.

254. 3200 lbs. per sq. in.

255. .27 in.

256. $d = .71$ in.; $h = .24$ in.

257. 13.6 ft.; 2050 lbs.-ft.

258. 16.61 in.^{3}

259. 5.56 in.; .83°.

261. 4 in.; 4.2 in.

262. Over right support, 16,000 lbs.-ft.

263. .064 in.

264. 146.2 H.P.

265. 5.84 in.

266. 3000 lbs.-ft.

267. .88 in.

268. .565 in.

269. $\dfrac{a(2l - 2b - a) + 2bl}{2l}$.

270. $\frac{3}{8}wl$, $\frac{5}{4}wl$, $\frac{3}{8}wl$.

271. .84 in.

272. 4.16 in., 1028 in.^4, 131.2 in.^3

273. 147 in.^3

274. 15.1 in., 8.5 in., 8950 lbs.

275. $\frac{3}{8}wl$.

276. 29,300 lbs.

INDEX